Problem 1

Redefinition of $\mathcal{L}$, so that h only appears in the following way: $\partial_\mu h^{\nu\rho}$:

$$\mathcal{L} = \frac{1}{2}\Bigg[\overbrace{(\partial_\mu h^{\mu\nu})(\partial_\nu \eta_{\rho\sigma} h^{\rho\sigma})}^{\mathcal{L}_1} - \overbrace{(\partial_\mu h^{\rho\sigma})(\partial_\rho \eta_{\sigma\nu} h^{\mu\nu})}^{\mathcal{L}_2} \tag{1}$$

$$+\frac{1}{2}\underbrace{\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})\left(\partial_\nu \eta_{\rho\xi}\eta_{\sigma\chi} h^{\xi\chi}\right)}_{\mathcal{L}_3} - \frac{1}{2}\underbrace{\eta^{\mu\nu}(\partial_\mu \eta_{\rho\sigma} h^{\rho\sigma})\left(\partial_\nu \eta_{\xi\chi} h^{\xi\chi}\right)}_{\mathcal{L}_4}\Bigg] \tag{2}$$

It's sufficient to examine the variation of the derivative of h. It holds:

$$\frac{\partial(\partial_\chi h^{\mu\nu})}{\partial(\partial_\alpha h^{\beta\gamma})} = \delta^\alpha_\chi \delta^\beta_\mu \delta\gamma_\nu \tag{3}$$

It follows:

$$\begin{aligned}\frac{\partial \mathcal{L}_1}{\partial(\partial_\alpha h^{\beta\gamma})} &= \delta^\alpha_\mu \delta^\beta_\mu \delta^\gamma_\nu\ \partial_\nu \eta_{\rho\sigma} h^{\rho\sigma} + \delta^\alpha_\nu \delta^\beta_\rho \delta^\gamma_\sigma\ \eta_{\rho\sigma}\partial_\mu h^{\mu\nu} &(4)\\ &= \delta^\alpha_\beta \partial_\gamma h + \eta_{\beta\gamma}\partial_\mu h^{\mu\alpha} &(5)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}_2}{\partial(\partial_\alpha h^{\beta\gamma})} &= \frac{1}{2}\left[\frac{\partial \mathcal{L}_2}{\partial(\partial_\alpha h^{\beta\gamma})} + \frac{\partial \mathcal{L}_2}{\partial(\partial_\alpha h^{\gamma\beta})}\right] &(6)\\ &= \frac{1}{2}\left[\delta^\alpha_\mu \delta^\beta_\rho \delta^\gamma_\sigma\ \partial_\rho \eta_{\sigma\nu} h^{\mu\nu} + \delta^\alpha_\rho \delta^\beta_\mu \delta^\gamma_\nu\ \eta_{\sigma\nu}\partial_\mu h^{\rho\sigma} + \delta^\alpha_\mu \delta^\gamma_\rho \delta^\beta_\sigma\ \partial_\rho \eta_{\sigma\nu} h^{\mu\nu} + \delta^\alpha_\rho \delta^\gamma_\mu \delta^\beta_\nu\ \eta_{\sigma\nu}\partial_\mu h^{\rho\sigma}\right] &(7)\\ &= \frac{1}{2}\left[\partial_\beta \eta_{\gamma\nu} h^{\alpha\nu} + \eta_{\sigma\gamma}\partial_\beta h^{\alpha\sigma} + \partial_\gamma \eta_{\beta\nu} h^{\alpha\nu} + \eta_{\sigma\beta}\partial_\gamma h^{\alpha\sigma}\right] = \eta_{\sigma\gamma}\partial_\beta h^{\alpha\sigma} + \eta_{\sigma\beta}\partial_\gamma h^{\alpha\sigma} &(8)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}_3}{\partial(\partial_\alpha h^{\beta\gamma})} &= \delta^\alpha_\mu \delta^\beta_\rho \delta^\gamma_\sigma\ \partial_\nu \eta^{\mu\nu}\eta_{\rho\xi}\eta_{\sigma\chi} h^{\xi\chi} + \delta^\alpha_\nu \delta^\beta_\xi \delta^\gamma_\chi\ \eta^{\mu\nu}\eta_{\rho\xi}\eta_{\sigma\chi}\partial_\mu h^{\rho\sigma} &(9)\\ &= \eta^{\alpha\nu}\partial_\nu h_{\beta\gamma} + \eta^{\mu\alpha}\partial_\mu h_{\beta\gamma} = 2\eta^{\alpha\nu}\partial_\nu h_{\beta\gamma} &(10)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h^{\beta\gamma})} &= \delta^\alpha_\mu \delta^\beta_\rho \delta^\gamma_\sigma\ \eta^{\mu\nu}\eta_{\rho\sigma}\partial_\nu h + \delta^\alpha_\nu \delta^\beta_\xi \delta^\gamma_\chi\ \eta^{\mu\nu}\eta_{\xi\chi}\partial_\mu h &(11)\\ &= \eta_{\beta\gamma}\eta^{\alpha\nu}\partial_\nu h + \eta_{\beta\gamma}\eta^{\mu\alpha}\partial_\mu h = 2\eta_{\beta\gamma}\eta^{\alpha\mu}\partial_\mu h &(12)\end{aligned}$$

Using the Lagrange formalism yields:

$$\begin{aligned}-0 &= \frac{\partial \mathcal{L}}{\partial h^{\beta\gamma}} - \frac{\partial}{\partial_\alpha}\frac{\partial \mathcal{L}}{\partial(\partial_\alpha h^{\beta\gamma})} &(13)\\ 0 &= \frac{1}{2}\left[\partial_\alpha \delta^\alpha_\beta \partial_\gamma h + \eta_{\beta\gamma}\partial_\mu h^{\mu\alpha} - \eta_{\sigma\gamma}\partial_\beta h^{\alpha\sigma} - \eta_{\sigma\beta}\partial_\gamma h^{\alpha\sigma} + \partial_\alpha \eta^{\alpha\nu}\partial_\nu h_{\beta\gamma} - \partial_\alpha \eta_{\beta\gamma}\eta^{\alpha\mu}\partial_\mu h\right] &(14)\\ &= \frac{1}{2}\left[\partial_\beta\partial_\gamma h + \partial_\alpha \eta_{\beta\gamma}\partial_\mu h^{\mu\alpha} - \partial_\beta\partial_\alpha h^\alpha{}_\gamma - \partial_\gamma\partial_\alpha h^\alpha{}_\beta + \underbrace{\eta^{\alpha\nu}\partial_\alpha\partial_\nu}_{\Box} h_{\beta\gamma} - \eta_{\beta\gamma}\underbrace{\eta^{\alpha\mu}\partial_\alpha\partial_\mu}_{\Box} h\right] &(15)\\ &= \frac{1}{2}\left[-\partial_\beta\partial_\alpha h^\alpha{}_\gamma - \partial_\gamma\partial_\alpha h^\alpha{}_\beta + \partial_\beta\partial_\gamma h + \Box h_{\beta\gamma} + \eta_{\beta\gamma}\partial_\alpha\partial_\mu h^{\mu\alpha} - \eta_{\beta\gamma}\Box h\right] = -G_{\beta\gamma} \quad \checkmark &(16)\end{aligned}$$

Problem 2

a) Starting with a general spherically symmetric metric:

$$ds_3^2 = \Phi(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{17}$$

One wants to write it in a new, isotropic, coordinate:

$$ds_3^2 = \Psi(\tilde{r})(\,\mathrm{d}\tilde{r}^2 + \tilde{r}^2\,\mathrm{d}\Omega^2) \tag{18}$$

Each component has to be equal:

$$\Phi(r)\,\mathrm{d}r^2 = \Psi(\tilde{r})\,\mathrm{d}\tilde{r}^2 \tag{19}$$

$$r^2 = \Psi(\tilde{r})\tilde{r}^2, \quad \Rightarrow \quad \Psi(\tilde{r}) = \frac{r^2}{\tilde{r}^2} \tag{20}$$

The last line gives:

$$\Phi(r)\,\mathrm{d}r^2 = \frac{r^2}{\tilde{r}^2}\,\mathrm{d}\tilde{r}^2, \quad \Rightarrow \quad \left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}\right)^2 \cdot \frac{1}{\tilde{r}^2} = \frac{\Phi(r)}{r^2} \tag{21}$$

There is an easy solution to this equation:

$$\log\tilde{r} = \int \frac{\sqrt{\Phi(r)}}{r}\,\mathrm{d}r \tag{22}$$

So this proves the problem as long as Φ and Ψ are smooth enough.

b) Inserting the coefficients of the Schwarzschild metric into above integral:

$$\tilde{r} = \exp\int \frac{1}{r\sqrt{1-\frac{2M}{r}}}\,\mathrm{d}r = C\left[2r\left(\sqrt{1-\frac{2M}{r}}+1\right) - 2M\right] \tag{23}$$

This gives

$$\Psi(r) = \frac{r^2}{\tilde{r}^2} = \frac{1}{16C^2}\left(1+\frac{2MC}{r}\right)^4 \tag{24}$$

Since there is no restriction for C we can choose $C = \frac{1}{4}$ which gives the result for $\mathrm{d}s_3^2$. Inserting $r(\tilde{r})$ into the first coefficient (Mathematica), yields the given result.

Problem 3

a) Assuming shell is made of dust: $T^{\mu\nu} = \rho U^\mu U^\nu, \rho = \rho_0\delta(R-r)$ with $\int_V \rho\,\mathrm{d}V = M \;\Rightarrow\; \rho_0 = \frac{M}{4\pi R^2}$. The coordinate for each particle on th shell is:

$$x^\mu = \begin{pmatrix} \lambda \\ R\cos(\varphi_0+\Omega\lambda)\sin\theta \\ R\sin(\varphi_0+\Omega\lambda)\sin\theta \\ R\cos\theta \end{pmatrix}, \quad \frac{\partial x^\mu}{\partial\lambda} = \begin{pmatrix} 1 \\ -R\Omega\sin(\varphi_0+\Omega\lambda)\sin\theta \\ R\Omega\cos(\varphi_0+\Omega\lambda)\sin\theta \\ 0 \end{pmatrix} = U^\mu \tag{25}$$

with $|U|^2 = 1+\frac{1}{2}(1-\cos 2\theta)R^2\Omega^2 \approx 1$, because $R\Omega \ll 1$. From this we get ($\sin\to S,\ \cos\to C,\ \varphi_0\equiv 0$ for truncation):

$$T^{\mu\nu} = \rho\begin{pmatrix} 1 & -R\Omega S(\theta)S(\lambda\Omega) & R\Omega C(\lambda\Omega)S(\theta) & 0 \\ -R\Omega S(\theta)S(\lambda\Omega) & R^2\Omega^2 S^2(\theta)S^2(\lambda\Omega) & -R^2\Omega^2 C(\lambda\Omega)S^2(\theta)S(\lambda\Omega) & 0 \\ R\Omega C(\lambda\Omega)S(\theta) & -R^2\Omega^2 C(\lambda\Omega)S^2(\theta)S(\lambda\Omega) & R^2\Omega^2 C^2(\lambda\Omega)S^2(\theta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

Dropping all terms with $R^2\Omega^2 \lll 1$ (full form):

$$T^{\mu\nu} = \rho\begin{pmatrix} 1 & -R\Omega\sin(\theta)\sin(\lambda\Omega+\varphi_0) & R\Omega\cos(\lambda\Omega+\varphi_0)\sin(\theta) & 0 \\ -R\Omega\sin(\theta)\sin(\lambda\Omega+\varphi_0) & 0 & 0 & 0 \\ R\Omega\cos(\lambda\Omega+\varphi_0)\sin(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{27}$$

Einstein equations:

$$G_{00} = 2\nabla^2\psi = 8\pi T_{00} = 8\pi \frac{M}{4\pi R^2}\delta(R-r) \tag{28}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + 2\overset{0}{\cancel{\partial_0}}\partial_j\psi = 8\pi T_{0j} \tag{29}$$

$$G_{ij} = \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\phi-\psi) - \overset{0}{\cancel{\partial_0}}\partial_{(i}w_{j)} + 2\delta_{ij}\overset{0}{\cancel{\partial_0^2}}\psi - \Box s_{ij} = 8\pi T_{ij} = 0 \tag{30}$$

Solving for ψ yields: $\psi(r) = \begin{cases} -\frac{M}{r}, & r \geq R \\ -\frac{M}{R}, & r < R \end{cases}$.
It is obvious that $\phi = \psi$, $w_3 = 0$, $s_{ij} = 0$. Solving for w_1, w_2 gives: [1]

$$w_1 = \begin{cases} -\frac{4M\Omega}{3R}\frac{R^2}{r^3}y, & r \geq R \\ -\frac{4M\Omega}{3R}y, & r \geq R \end{cases}, \tag{31}$$

$$w_2 = \begin{cases} \frac{4M\Omega}{3R}\frac{R^2}{r^3}y, & r \geq R \\ \frac{4M\Omega}{3R}y, & r \geq R \end{cases} \tag{32}$$

Based on this for the inside of the shell, the gravito-electric field vanishes, because $G_i = -\partial_i\phi - \partial_0 w_i = 0$, for the gravito-magnetic field one obtains: $H^i = (\nabla \times \vec{w})^i = -\frac{8M\Omega}{3R}\mathbf{e}_z$

b) The momentum of the observer be $p = mv \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix}$, with $v \ll c$ and E its energy:

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = E(G^i + (\nabla \times \vec{w})^i - \dots] = Ev \begin{pmatrix} \cos\varphi H_z \\ 0 \\ 0 \end{pmatrix} = -Ev_y\frac{8M\Omega}{3R}\mathbf{e}_x \tag{33}$$

Which shows that only the radial component adds to the effect, of the radial motion.

[1] http://aforrester.bol.ucla.edu/educate/Articles/Derive_GravitoEM.pdf