Problem 1

a) Starting with the general line element:

$$\mathrm{d}s^2 = -\mathrm{e}^{2\alpha(r)}\,\mathrm{d}t^2 + \mathrm{e}^{2\beta(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \tag{1}$$

which is equivalent to the metrik $g_{\mu\nu} = \mathrm{diag}(-\mathrm{e}^{2\alpha(r)}, \mathrm{e}^{2\beta(r)}, r^2)$, from that one obtains with Mathematica the Christoffel Symbols, the Riemann Tensor, the Ricci Tensor and the Ricci Scalar:

$$R_{\mu\nu} = \mathrm{diag}\left(e^{2\alpha(r)-2\beta(r)}\left(\alpha''(r) + \alpha'(r)\left(\alpha'(r) - \beta'(r) + \frac{1}{r}\right)\right),\right. \tag{2}$$

$$-\alpha''(r) + \alpha'(r)\beta'(r) - \alpha'(r)^2 + \frac{\beta'(r)}{r}, \tag{3}$$

$$\left. r\left(-e^{-2\beta(r)}\right)\left(\alpha'(r) - \beta'(r)\right)\right), \tag{4}$$

$$R = \frac{e^{-2\beta(r)}\left(2r\alpha''(r) + 2\left(r\alpha'(r)+1\right)\left(\alpha'(r) - \beta'(r)\right)\right)}{r} \tag{5}$$

The Energy Momentum Tensor is given as:

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + p g_{\mu\nu} \tag{6}$$

The velocity is normalised with $u_\mu = \left(-\sqrt{|g_{11}|}, 0, 0\right)$ so the energy momentum tensor will be:

$$T_{\mu\nu} = \mathrm{diag}\left(\rho\mathrm{e}^{2\alpha}, p\mathrm{e}^{2\beta}, \frac{p}{r^2}\right) \tag{7}$$

Inserting this into the Einstein Equation delivers:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa T_{\mu\nu} \tag{8}$$

This gives the two equations for α and β:

$$\alpha' = \kappa r p \mathrm{e}^{2\beta}, \qquad \beta' = \kappa r p \mathrm{e}^{2\beta} \tag{9}$$

The second equation is equivalent to $m' = 2\pi r\rho$ with $m = \frac{1}{8}\left(1 - \mathrm{e}^{-2\beta}\right)$:

$$m(r) = 2\pi\int_0 (r) r'\rho(r')\,\mathrm{d}r', \qquad m(r=0) \overset{!}{=} 0 \tag{10}$$

With the help of mathematica the conservation of energy $\nabla_\mu T_{\mu\nu} = 0$ delivers:

$$0 = \mathrm{e}^{-2\beta}\left(\alpha'(p+\rho) + p'\right) \tag{11}$$

this gives the TOV equation:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -(p+\rho)\frac{8\pi r p}{1-8m} \tag{12}$$

b) In vaccum, the einstein equations vanish: $G_{\mu\nu} = 0$ and so $\frac{\alpha'}{r} = \frac{\beta'}{r} = 0$ so they are not a function of r. By choosing $\alpha = 0 \to \mathrm{e}^{2\alpha} = 1$ and $\beta = -\frac{1}{2}\log\left(1 - 8GM\right) \to \mathrm{e}^{2\beta} = \frac{1}{1-8GM}$ gives the desired result.

c) The result can be obtained quite obvious by the following substitutions:

$$\tau := t, \qquad \xi := \frac{r}{\sqrt{1-8GM}}, \qquad \phi := \sqrt{1-8GM}\,\vartheta \tag{13}$$

d) Setting $\rho(r) = \rho_0$ gives $m(r) = \pi r^2 \rho_0$. Inserting this into the TOV Equation gives:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -(p+\rho)\frac{8\pi r p}{1 - 8\pi r^2 \rho_0} \tag{14}$$

By seperating the variables and integrating one obtains:

$$p(r) = \rho_0 \frac{A\sqrt{1-8\pi r^2\rho_0}}{1 - A\sqrt{1-8\pi r^2\rho_0}}, \quad A = \text{const.} \tag{15}$$

Solving the equation for α' gives:

$$\alpha(r) = \log\left(1 - A\sqrt{1-8\pi r^2\rho_0}\right) + B, \quad B = \text{const.} \tag{16}$$

So finally the metric can be written as:

$$\mathrm{d}s^2 = -\left(1 - A\sqrt{1-8\pi r^2\rho_0}\right)^2 \mathrm{d}t^2 + \frac{1}{1-8\pi r^2\rho_0}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\vartheta^2 \tag{17}$$

The proper mass is given as:

$$M(R) = 2\pi \int_0^R \rho_0 \frac{r}{\sqrt{1-8\pi r^2\rho_0}}\,\mathrm{d}r = \frac{1}{4}\left[1 - \sqrt{1-8\pi R^2\rho_0}\right] \tag{18}$$

e) Now: $p = \kappa\rho^{2/3}$ which can be easily inserted in the TOV equation. Solving this requires some higher level of magic but I am out of mana...