Problem 1

Inserting the definition: $\partial_\mu \bar{h}^{\mu\nu} = \partial_\mu h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu h = 0$. Also: $\Gamma^\rho_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right)$:

$$\Box x^\chi = g^{\mu\nu}\nabla_\mu\nabla_\nu x^\chi = g^{\mu\nu}(\partial_\mu - \Gamma^\rho_{\mu\nu})\partial_\rho x^\chi = -g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0 \tag{1}$$

$$= -\frac{1}{2}\eta^{\mu\nu}\eta^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) + \mathcal{O}(h^2) \tag{2}$$

$$= \frac{1}{2}\partial_\lambda h - \partial_\mu h^{\mu\rho} = \partial_\mu \bar{h}^{\mu\nu} \quad \checkmark \tag{3}$$

Problem 2

a) For the movement along the x-axis the acceleration is given by: $m\ddot{x} = -\frac{m^2}{|x_1 - x_2|^2}$. Taking symmetry into account ($x_1 = -x_2$) this can be rewritten and solved with the polynomial approach $x(t) = at^n$:

$$\ddot{x} = -\frac{m}{4x^2} \tag{4}$$

$$n(n-1)at^{n-2} = -\frac{m}{4}(at^n)^{-2} \quad \Rightarrow x(t) = \pm\left(\frac{9m}{8}t^2\right)^{1/3} \tag{5}$$

b) This approximation is reasonable as long as $|h| \ll 1$ holds and the connection is negligible.

c) Using the ansatz from the lecture:

$$\bar{h}_{ij}(t, \vec{x}) = \frac{2}{r}\frac{\mathrm{d}^2}{\mathrm{d}t^2}I_{ij}(t_r), \quad I_{ij} = \iint y^i y^j T^{00}, \tag{6}$$

$$T^{00} = \delta(z)\delta(y)(\delta(x - (9mt^2/8)^{1/3}) + \delta(x + (9mt^2/8)^{1/3}))m \tag{7}$$

This yields:

$$h_{1j} = h_{i1} = h_{ij} \equiv 0, \quad i, j = \{2, 3\}, \tag{8}$$

$$h_{11}(t, \vec{r}) = \frac{4}{r}\frac{\mathrm{d}^2}{\mathrm{d}t_r^2}x^2(t) = -\frac{16}{9r}(9m/(8t_r))^{2/3}, \quad t_r = |t - x(t)|, \quad r = R \tag{9}$$