Problem 1

The space monkey got up to two possible orbits: A stable one and an instable one. The instable one is closer to the black hole.
If the space monkey catches the coconut while he is on the instable orbit he will inevitably fall into the black hole.
If he sits on the stable orbit there are three more options:

- The coconut is fast enough that it fires the monkey over the peak of the potential into the black hole
- The coconut is that fast the he doesn't reach the peak, but it's energy is enough to escape the black hole after 'sliding down the wall'
- The energy is that low that he remain roughly in his orbit which is now deformed to an elliptical orbit

Problem 2

Using the Schwarzschild-Metrik the normalised four-velocity is defined as:

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -1 \tag{1}$$

$$-\left(1-\frac{2GM}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2\sin^2\theta\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = -1 \tag{2}$$

For $r < 2GM$ only the second term is negative, so we get:

$$\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \leq -1 \tag{3}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \geq \left(\frac{2GM}{r}-1\right) \tag{4}$$

$$\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| \geq \sqrt{\frac{2GM}{r}-1} \tag{5}$$

Maximum lifetime:

$$\mathrm{d}\tau = \frac{\mathrm{d}r}{\mathrm{d}\tau}\,\mathrm{d}r \tag{6}$$

$$\tau_{\max} = \int_{2GM}^{0}\frac{\mathrm{d}r}{\mathrm{d}\tau}\,\mathrm{d}r \tag{7}$$

$$\tau_{\max} = \int_{2GM}^{0}\frac{1}{\sqrt{\frac{2GM}{r}-1}}\,\mathrm{d}r \tag{8}$$

$$\text{...Mathematica...} \tag{9}$$

$$\tau_{\max} = \pi GM \tag{10}$$

Inserting the proper factors on gets:

$$\tau_{\max} = 1.56\cdot 10^{-5}\frac{\mathrm{s}}{\mathrm{solarmass}} \tag{11}$$