Problem 1

Set $m = 1$.
For a n-dimensional distribution Φ and a function F on a manifold we get:

$$\int_{\mathbb{R}^n} F(x)\partial_\mu \Phi(x)\,\mathrm{d}^n x = -\int_{\mathbb{R}^n} \Phi(x)\partial_\mu F(x)\,\mathrm{d}^n x \tag{1}$$

Starting with:

$$0 = \Delta_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} \tag{2}$$

$$= \int \left[\frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{|g(x(\tau))|}} \partial_{y^\mu} \delta(y - x(\tau)) + \frac{\delta(y - x(\tau)}{\sqrt{|g(x(\tau))|}} \left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right) \right] \mathrm{d}\tau \tag{3}$$

For an arbitrary Function F we get:

$$0 = \int_M \left(\Delta_\mu T^{\mu\nu}\right)\big|_y F(y)\sqrt{|g(y))|}\,\mathrm{d}y \tag{4}$$

$$= \int_M \int \left[\frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{|g(x(\tau))|}} \partial_{y^\mu} \delta(y - x(\tau)) + \frac{\delta(y - x(\tau)}{\sqrt{|g(x(\tau))|}} \left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right) \right] \mathrm{d}\tau F(y)\sqrt{|g(y))|}\,\mathrm{d}y \tag{5}$$

$$= \int \frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{|g(x(\tau))|}} \int_M F(y)\sqrt{|g(y)|}\partial_{y^\mu}\delta(y - x(\tau))\,\mathrm{d}y\,\mathrm{d}\tau \tag{6}$$

$$+ \int \frac{\left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right)}{\sqrt{|g(x(\tau))|}} \int_M \delta(y - x(\tau))F(y)\sqrt{|g(y)|}\,\mathrm{d}y\,\mathrm{d}\tau \tag{7}$$

$$= \int \left[-\frac{\dot{x}^\nu}{\sqrt{|g(x(\tau))|}} \dot{x}^\mu \partial_\mu \left(F(x(\tau))\sqrt{|g(x(\tau))|} \right) + \frac{\left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right)}{\sqrt{|g(x(\tau))|}} F(x(\tau))\sqrt{|g(x(\tau))|} \right] \mathrm{d}\tau \tag{8}$$

$$= \int F(x(\tau)) \left[\sqrt{|g(x(\tau))|} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\dot{x}^\nu}{\sqrt{|g(x(\tau))|}} \right) + \left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right) \right] \mathrm{d}\tau \tag{9}$$

$$= \int F(x(\tau)) \left[\ddot{x} + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda + \dot{x}^\nu \sqrt{|g(x(\tau))|} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{1}{\sqrt{|g(x(\tau))|}} \right) + \Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu \right] \mathrm{d}\tau \tag{10}$$

$$= \int F(x(\tau)) \left[\ddot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda - \dot{x}^\nu \dot{x}^\lambda \Gamma^\mu_{\mu\lambda} + \dot{x}^\nu \dot{x}^\lambda \Gamma^\mu_{\mu\lambda} \right] \mathrm{d}\tau \tag{11}$$

$$= \int F(x(\tau)) \left[\ddot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right] \mathrm{d}\tau = 0 \tag{12}$$

So we can conclude that $x(\tau)$ has to solve the geodetic equation.

Problem 2

Be $\tilde{g}_{\mu\nu}$ the old metric and $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$ the new one ($g^{\mu\nu} = \Omega^{-2}\tilde{g}^{\mu\nu}$). So one has to calculate $\Gamma = \Gamma(\tilde{\Gamma}, \Omega), R^\rho_{\sigma\mu\nu} = R^\rho_{\sigma\mu\nu}(\tilde{R}^\rho_{\sigma\mu\nu}, \Omega), R_{\mu\nu} = R_{\mu\nu}(\tilde{R}_{\mu\nu}, \Omega), R = R(\tilde{R}, \Omega)$:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right) \tag{13}$$

$$= \frac{\Omega^{-2}}{2} \tilde{g}^{\rho\lambda} \left(\partial_\mu \Omega^2 \tilde{g}_{\lambda\nu} + \partial_\nu \Omega^2 \tilde{g}_{\lambda\mu} - \partial_\lambda \Omega^2 \tilde{g}_{\mu\nu} \right) \tag{14}$$

$$= \underbrace{\frac{\Omega^{-2}}{2} \tilde{g}^{\rho\lambda} \left(\partial_\mu \tilde{g}_{\lambda\nu} + \partial_\nu \tilde{g}_{\lambda\mu} - \partial_\lambda \tilde{g}_{\mu\nu} \right)}_{\tilde{\Gamma}^\rho_{\mu\nu}} + \frac{\Omega^{-2}}{2} \tilde{g}^{\rho\lambda} \left(\tilde{g}_{\lambda\nu} \partial_\mu \Omega^2 \tilde{+} \tilde{g}_{\lambda\mu} \partial_\nu \Omega^2 - \tilde{g}_{\mu\nu} \partial_\lambda \Omega^2 \right) \tag{15}$$

$$\tag{16}$$

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu} \tag{17}$$

$$= \tilde{R}^{\rho}{}_{\sigma\mu\nu} \tag{18}$$

$$+ \partial_\mu \left(\frac{\Omega^{-2}}{2} \tilde{g}^{\rho\lambda} \left(\tilde{g}_{\lambda\nu}\partial_\mu \Omega^2 \tilde{+} \tilde{g}_{\lambda\mu}\partial_\nu \Omega^2 - \tilde{g}_{\mu\nu}\partial_\lambda \Omega^2 \right) \right) \tag{19}$$

$$+ \dots \tag{20}$$

$$\overset{\text{Mathematica}}{=} \tilde{R}^{\rho}{}_{\sigma\mu\nu} \tag{21}$$

$$+ \frac{\Omega^{-2}}{2} \left(\delta^{\rho}_{\nu}\partial_\mu\partial_\sigma\Omega^2 - \delta^{\rho}_{\mu}\partial_\nu\partial_\sigma\Omega^2 + g_{\sigma\mu}\partial_\nu \left(g^{\rho\lambda}\partial_\lambda\Omega^2 \right) - g_{\sigma\nu}\partial_\mu \left(g^{\rho\lambda}\lambda\Omega^2 \right) \right) \tag{22}$$

$$+ \frac{\Omega^{-2}}{2} \left(g^{\kappa\lambda} \left(\partial_\lambda \Omega^2 \right) \left(\Gamma^{\rho}_{\nu\kappa} g_{\sigma\mu} - \Gamma^{\rho}_{\mu\kappa} g_{\sigma\nu} \right) + \left(\partial_\kappa \Omega^2 \right) \left(\Gamma^{\kappa}_{\sigma\nu}\delta^{\rho}_{\mu} - \Gamma^{\kappa}_{\sigma\mu}\delta^{\rho}_{\nu} \right) \right) \tag{23}$$

$$+ \left(\frac{\Omega^{-2}}{2} \right)^2 \left[\left(\partial_\sigma \Omega^2 \right) \left(\delta^{\rho}_{\mu} \left(\partial_\nu \Omega^2 \right) - \delta^{\rho}_{\nu} \left(\partial_\mu \Omega^2 \right) \right) + g^{\kappa\lambda} \left(\partial_\kappa \Omega^2 \right) \left(\partial_\lambda \Omega^2 \right) \left(\delta^{\rho}_{\nu} g_{\mu\sigma} - \delta^{\rho}_{\mu} g_{\sigma\nu} \right) \right. \tag{24}$$

$$\left. + g^{\rho\lambda} \left(\partial_\lambda \Omega^2 \right) \left(g_{\nu\sigma} \left(\partial^2_\Omega \right) - g_{\mu\sigma} \left(\partial_\nu \Omega^2 \right) \right) \right] \tag{25}$$

The Ricci Tensor follows as:

$$R_{\sigma\nu} = \tilde{R}_{\sigma\nu} \tag{26}$$

$$+ \frac{\Omega^{-2}}{2} \left[(2-n)\, \partial_\sigma\partial_\nu\Omega^2 - \partial_\nu \left(g^{\mu\lambda} g_{\sigma\nu} \partial_\lambda \Omega^2 \right) \right] \tag{27}$$

$$+ \frac{\Omega^{-2}}{2} \left[g^{\kappa\lambda} \left(\partial_\lambda \Omega^2 \right) \left[\Gamma^{\mu}_{\nu\kappa} g_{\sigma\mu} + \Gamma^{\mu}_{\sigma\kappa} g_{\nu\mu} - \Gamma^{\mu}_{\mu\kappa} g_{\sigma\nu} \right] + (n-2) \left(\partial_\kappa \Omega^2 \right) \Gamma^{\kappa}_{\sigma\nu} \right] \tag{28}$$

$$+ \left(\frac{\Omega^{-2}}{2} \right)^2 \left[\left(\partial_\sigma \Omega^2 \right) \left(\partial_\nu \Omega^2 \right) - g^{\mu\lambda} g_{\sigma\nu} \left(\partial_\lambda \Omega^2 \right) \left(\partial_\mu \Omega^2 \right) \right] \tag{29}$$

The Ricci Scalar is:

$$R = \Omega^{-2}\tilde{R} + \frac{\Omega^{-4}}{2} \left[(n-2) g^{\nu\sigma} \Gamma^{\kappa}_{\sigma\nu} \partial_\kappa \Omega^2 - 2(n-1) g^{\nu\sigma} \partial_\sigma\partial_\nu \Omega^2 - n g^{\kappa\lambda} \left(\partial_\lambda \Omega^2 \right) \Gamma^{\mu}_{\mu\kappa} \right] \tag{30}$$

$$- \frac{\Omega^{-3}}{4} (n-1)(n-2) g^{\nu\sigma} \left(\partial_\sigma \Omega^2 \right) \left(\partial_\nu \Omega^2 \right) \tag{31}$$

$\tilde{g}_{\mu\nu}$ can be choosen arbitrarily so we can set it to $g_{\mu\nu} = \eta_{\mu\nu}$ with vanishing derivatives and vanishing christoffel symbols. Inserting the above results in the definition of the Weyl tensor gives:

$$C^{\rho}{}_{\sigma\mu\nu} = \dots = \tilde{C}^{\rho}{}_{\sigma\mu\nu} \tag{32}$$