Problem 1

a) The metric is given as:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2(\psi) & 0 \\ 0 & 0 & \sin^2(\theta)\sin^2(\psi) \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \csc^2(\psi) & 0 \\ 0 & 0 & \csc^2(\theta)\csc^2(\psi) \end{pmatrix} \tag{1}$$

With the inverse metric $g^{\mu\nu}$ and $\csc = \frac{1}{\sin}$. From this we get directly the Christoffel Symbols:

$$\Gamma^0{}_{11} = -\cos(\psi)\sin(\psi) \tag{2}$$
$$\Gamma^0{}_{22} = -\cos(\psi)\sin^2(\theta)\sin(\psi) \tag{3}$$
$$\Gamma^1{}_{01} = \cot(\psi) = \Gamma^1{}_{10} \tag{4}$$
$$\Gamma^1{}_{22} = -\cos(\theta)\sin(\theta) \tag{5}$$
$$\Gamma^2{}_{02} = \cot(\psi) = \Gamma^2{}_{20} \tag{6}$$
$$\Gamma^2{}_{12} = \cot(\theta) = \Gamma^2{}_{21} \tag{7}$$

All other symbols vanish.

b) From the above we get directly the riemann tensor:

$$\mathrm{R}^0{}_{110} = \sin^2(\psi) \qquad = -\mathrm{R}^0{}_{101} \tag{8}$$
$$\mathrm{R}^0{}_{220} = \sin^2(\theta)\sin^2(\psi) \qquad = -\mathrm{R}^0{}_{202} \tag{9}$$
$$\mathrm{R}^1{}_{010} = \cot^2(\psi) - \csc^2(\psi) \qquad = -\mathrm{R}^1{}_{001} \tag{10}$$
$$\mathrm{R}^1{}_{221} = \sin^2(\theta) - \cos^2(\psi)\sin^2(\theta) \qquad = -\mathrm{R}^1{}_{212} \tag{11}$$
$$\mathrm{R}^2{}_{020} = \cot^2(\psi) - \csc^2(\psi) \qquad = -\mathrm{R}^2{}_{002} \tag{12}$$
$$\mathrm{R}^2{}_{121} = \cos^2(\psi) + \cot^2(\theta) - \csc^2(\theta) \qquad = -\mathrm{R}^2{}_{112} \tag{13}$$

The Ricci Tensor immediately follows as:

$$R_{\mu\nu} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2\sin^2(\psi) & 0 \\ 0 & 0 & -2\sin^2(\theta)\sin^2(\psi) \end{pmatrix} \tag{14}$$

As well as the Ricci Scalar:

$$R = -6 \tag{15}$$

c) Proof by calculating it ($R_{\rho\sigma\mu\nu} = g_{\rho\tau}R^{\tau}_{\sigma\mu\nu}$):

$$\text{Table}\left[\text{Rf}_{\rho,\sigma,\mu,\nu} - \frac{\text{Rs}(G[[\rho,\mu]]G[[\sigma,\nu]] - G[[\sigma,\mu]]G[[\rho,\nu]])}{n(n-1)}, \{\rho,1,3\}, \{\sigma,1,3\}, \{\mu,1,3\}, \{\nu,1,3\}\right] = \tag{16}$$

$$\begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \end{pmatrix} \qquad \checkmark \tag{17}$$