Problem 1

a) We know that

$$g^{\mu\nu}g_{\mu\lambda} = \delta^{\nu}_{\lambda} \tag{1}$$

We already proved that $\nabla_\sigma g_{\mu\nu} =$, so it follows:

$$\nabla_\sigma(g^{\mu\nu}g_{\mu\lambda}) = \nabla_\sigma \delta^{\nu}_{\lambda} \tag{2}$$
$$g_{\mu\lambda}\nabla_\sigma g^{\mu\nu} + g^{\mu\nu}\nabla_\sigma g_{\mu\lambda} = 0 \tag{3}$$
$$g_{\mu\lambda}\nabla_\sigma g^{\mu\nu} = 0 \qquad \Rightarrow \qquad \nabla_\sigma g^{\mu\nu} = 0 \quad \checkmark \tag{4}$$

b) With the above result it is easier to show this part:

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = C, \qquad C = \text{const.} \tag{5}$$
$$\nabla_\lambda(\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma}) = 0 \tag{6}$$
$$\epsilon_{\mu\nu\rho\sigma}\nabla_\lambda(\epsilon^{\mu\nu\rho\sigma}) + \epsilon^{\mu\nu\rho\sigma}\nabla_\lambda(\epsilon_{\mu\nu\rho\sigma}) = 0 \tag{7}$$
$$2\epsilon^{\mu\nu\rho\sigma}\nabla_\lambda(\epsilon_{\mu\nu\rho\sigma}) = 0 \quad \checkmark \tag{8}$$

The last line follows from the one above by lowering the indices in the first summand, which will give us $\epsilon^{\mu'\nu'\rho'\sigma'} = g^{\mu'\mu}g^{\nu'\nu}\cdots\epsilon_{\mu\nu\rho\sigma}$. Using the product rule and the result from above allows us get the metric in front of the nabla symbol where it drags the indices up.

Problem 2

$g_{\mu\nu} \propto \delta_{\mu\nu}$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right] \tag{9}$$
$$= \frac{1}{2}g^{\lambda\lambda}\left[\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda}\right] = 0 \tag{10}$$

$$\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2}g^{\lambda\rho}\left[\partial_\mu g_{\mu\rho} + \partial_\mu g_{\mu\rho} - \partial_\rho g_{\mu\mu}\right] \tag{11}$$
$$= \frac{1}{2}g^{\lambda\lambda}\left[2\partial_\mu g_{\mu\lambda} - \partial_\lambda g_{\mu\mu}\right] \tag{12}$$
$$= -\frac{1}{2}g^{\lambda\lambda}\partial_\lambda g_{\mu\mu} \tag{13}$$

$$\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}g^{\lambda\rho}\left[\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}\right] \tag{14}$$
$$= \frac{1}{2}g^{\lambda\lambda}\left[\partial_\mu g_{\lambda\lambda} + \partial_\lambda g_{\mu\lambda} - \partial_\lambda g_{\mu\lambda}\right] \tag{15}$$
$$= \frac{1}{2}g^{\lambda\lambda}\partial_\mu g_{\lambda\lambda} \tag{16}$$
$$= \partial_\mu\left(\log\sqrt{|g_{\lambda\lambda}|}\right) \tag{17}$$

$$\Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2}g^{\lambda\rho}\left[\partial_\lambda g_{\lambda\rho} + \partial_\lambda g_{\lambda\rho} - \partial_\rho g_{\lambda\lambda}\right] \tag{18}$$
$$= \frac{1}{2}g^{\lambda\lambda}\left[\partial_\lambda g_{\lambda\lambda} + \partial_\lambda g_{\lambda\lambda} - \partial_\lambda g_{\lambda\lambda}\right] \tag{19}$$
$$= \partial_\lambda\left(\log\sqrt{|g_{\lambda\lambda}|}\right) \tag{20}$$

Problem 3

a)

$$J = \frac{\partial x^\mu}{\partial x^{\mu'}} = \begin{pmatrix} v\cos\phi & u\cos\phi & -uv\sin\phi \\ v\sin\phi & u\sin\phi & uv\cos\phi \\ u & -v & 0 \end{pmatrix} \qquad J^{-1} = \frac{\partial x^{\mu'}}{\partial x^\mu} = \begin{pmatrix} \frac{v\cos(\phi)}{u^2+v^2} & \frac{v\sin(\phi)}{u^2+v^2} & \frac{u}{u^2+v^2} \\ \frac{u\cos(\phi)}{u^2+v^2} & \frac{u\sin(\phi)}{u^2+v^2} & -\frac{v}{u^2+v^2} \\ -\frac{\sin(\phi)}{\mathrm{uv}} & \frac{\cos(\phi)}{\mathrm{uv}} & 0 \end{pmatrix} \tag{21}$$

Singular points for $u = 0$ or $v = 0$ (compare with J^{-1}).

b) A new coordinae basis in terms of the old one is obtained by multiplying the inverse of the transformation. The oneforms transform the opposite way:

$$\partial_{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\partial_\mu, \qquad \omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\omega_\mu \tag{22}$$

So the results are the lines in J^{-1} and J.

c) The line element must be constant in every coordinate system, this is analog to $v^\intercal g w = (v')^\intercal g' w'$ (paraboloidal coordinates with apostrophe, cartesian without). From that we get:

$$v' = Jv \quad \Rightarrow \quad v^\intercal g w = (Jv)^\intercal g W^\intercal = v^\intercal J^\intercal g J w \quad \Rightarrow \quad g' = J^\intercal g J = J^\intercal J \tag{23}$$

$$\Rightarrow g'_{\mu\nu} = \begin{pmatrix} u^2+v^2 & 0 & 0 \\ 0 & u^2+v^2 & 0 \\ 0 & 0 & u^2v^2 \end{pmatrix} \qquad g'^{\mu\nu} = \begin{pmatrix} \frac{1}{u^2+v^2} & 0 & 0 \\ 0 & \frac{1}{u^2+v^2} & 0 \\ 0 & 0 & \frac{1}{u^2v^2} \end{pmatrix} \tag{24}$$

and also $|g| = (u^2+v^2)^2u^2v^2$

d)

$$\Gamma^1_{11} = \frac{u}{u^2+v^2}, \qquad \Gamma^1_{12} = \Gamma^1_{21} = \frac{v}{u^2+v^2}, \qquad \Gamma^1_{13} = \Gamma^1_{31} = 0, \tag{25}$$

$$\Gamma^1_{22} = \frac{-u}{u^2+v^2}, \qquad \Gamma^1_{23} = \Gamma^1_{32} = 0, \qquad \Gamma^1_{33} = \frac{-uv^2}{u^2+v^2} \tag{26}$$

$$\Gamma^2_{11} = \frac{-v}{u^2+v^2}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{u}{u^2+v^2}, \qquad \Gamma^2_{13} = \Gamma^2_{31} = 0, \tag{27}$$

$$\Gamma^2_{22} = \frac{v}{u^2+v^2}, \qquad \Gamma^2_{23} = \Gamma^2_{32} = 0, \qquad \Gamma^2_{33} = \frac{-u^2v}{u^2+v^2} \tag{28}$$

$$\Gamma^3_{11} = 0, \qquad \Gamma^3_{12} = \Gamma^3_{21} = 0, \qquad \Gamma^3_{13} = \Gamma^3_{31} = 0, \tag{29}$$

$$\Gamma^3_{22} = 0, \qquad \Gamma^3_{23} = \Gamma^3_{32} = 0, \qquad \Gamma^3_{33} = 0 \tag{30}$$

e)

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda}V^\lambda = \partial_u V^u + \partial_v V^v + \partial_\phi V^\phi + \frac{uV^u + vV^v + uV^u + vV^v}{u^2+v^2} \tag{31}$$

$$= \partial_u V^u + \partial_v V^v + \partial_\phi V^\phi + 2\frac{uV^u + vV^v}{u^2+v^2} \tag{32}$$

The laplacian can be written in the following way (compare http://en.wikipedia.org/wiki/List_of_formulas_in_Riemannian_geometry#Gradient.2C_divergence.2C_Laplace.E2.80.93Beltrami_operator)

$$\Delta V = g^{\mu\nu}\frac{\partial^2 f}{\partial x^\mu \partial x^\nu} - g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\frac{\partial f}{\partial x^\lambda} = \frac{1}{u^2+v^2}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) + \frac{1}{u^2v^2}\frac{\partial^2 f}{\partial \phi^2} \tag{33}$$

$$- \frac{1}{(u^2+v^2)^2}\left(u\cancel{\frac{\partial f}{\partial u}} - v\cancel{\frac{\partial f}{\partial v}} - u\cancel{\frac{\partial f}{\partial u}} + v\cancel{\frac{\partial f}{\partial v}} - uv^2\frac{\partial f}{\partial u} - u^2v\frac{\partial f}{\partial v}\right) \tag{34}$$