Problem 2

For two vector fields X and Y and two functions f, g and two numbers a, b the commutator is defined as and the component formula for a vector field and a function is:

$$[X,Y](f) := X(Y(f)) - Y(X(f)), \qquad X(f) = X^\mu \partial_\mu f \tag{1}$$

From which obviously follows the Leibnitz Formula $X(fg) = X^\mu \partial_m u(fg) = fX^\mu \partial_\mu g + gX^\mu \partial_\mu f$. Linearity also follows. Its properties follow from:

$$[X,Y](af+bg] = X(Y(af+bg)) - Y(X(af+bg)) = X(aY(f)+b(Y(g)) - Y(aX(f)+bX(g)) \tag{2}$$

$$= aX(Y(f)) + bX(Y(g)) - aY(X(f)) - bY(X(g)) \tag{3}$$

$$= a[X,Y](f) + b[X,Y](g) \tag{4}$$

$$[X,Y](fg) = X(Y(fg)) - Y(X(fg)) = X(fYg + gYf) - Y(fXg + gXf) = \tag{5}$$

$$= X(f(Y(g))) + X(g(Y(f))) - Y(f(X(g))) - Y(g(X(f))) \tag{6}$$

$$= fX(Y(g)) + Y(g)X(f) + gX(Y(f)) + Y(f)X(g) \tag{7}$$

$$- fY(X(g)) - X(g)Y(f) - gY(X(f)) - X(f)Y(g) \tag{8}$$

$$= fX(Y(g)) + gX(Y(f)) - fY(X(g)) - gY(X(f)) \tag{9}$$

$$= f[X,Y](g) + g[X,Y](f) \tag{10}$$

$$([X,Y](f))^\nu = X^\mu \partial_\mu (Y^\nu \partial_\nu f) - Y^\mu \partial_\mu (X^\nu \partial_\nu f) \tag{11}$$

$$= X^\mu (\partial_\mu Y^\nu \partial_\nu f + Y^\nu \partial_\mu \partial_\nu f) - Y^\mu (\partial_\mu X^\nu \partial_\nu f + X^\nu \partial_\mu \partial_\nu f) \tag{12}$$

$$= X^\mu (\partial_\mu Y^\nu \partial_\nu f) - Y^\mu (\partial_\mu X^\nu \partial_\nu f) \tag{13}$$

$$\Rightarrow ([X,Y])^\nu = X^\mu (\partial_\mu Y^\nu) - Y^\mu (\partial_\mu X^\nu) \tag{14}$$

Change ALL the indices! (15)

$$([X,Y])^{\nu'} = X^{\mu'} \partial_{\mu'} Y^{\nu'} - Y^{\mu'} \partial_{\mu'} X^{\nu'} \tag{16}$$

$$= \frac{\partial x^{\mu'}}{\partial x^\mu} X^\mu \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^{\nu'}}{\partial x^\nu} Y^\nu\right) - Y^\mu \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^{\nu'}}{\partial x^\nu} X^\nu\right) \tag{17}$$

$$\text{first summands} \quad = \underbrace{\frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^{\mu'}}}_{1} X^\mu \left(\frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\mu} Y^\nu + \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu Y^\nu\right) \tag{18}$$

$$\text{cancel eachother} \quad - Y^\mu \left(\frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\mu} X^\nu + \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu X^\nu\right) \tag{19}$$

(20)

$$= \frac{\partial x^{\nu'}}{\partial x^\nu} X^\mu \partial_\mu Y^\nu - \frac{\partial x^{\nu'}}{\partial x^\nu} Y^\mu \partial_\mu X^\nu = \frac{\partial x^{\nu'}}{\partial x^\nu} ([X,Y])^\mu \tag{21}$$

Problem 1

R is the distance from the center of the tube to the center of the torus, r is the radius of the tube, from this we get an atlas made of four maps:

$$\Phi_1 : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3, \qquad \Phi_2 : (\pi, 3\pi) \times (0, 2\pi) \to \mathbb{R}^3, \tag{22}$$

$$\Phi_2 : (\pi, 3\pi) \times (0, 2\pi) \to \mathbb{R}^3, \qquad \Phi_4 : (\pi, 3\pi) \times (\pi, 3\pi) \to \mathbb{R}^3 \tag{23}$$