Aufgabe 27

a)

Da $H(x) = \frac{1}{2m}\sum_i p_i^2 \geq 0 \rightarrow E \geq 0$

$$\begin{aligned} Z_\beta &= \frac{1}{h^{3N}N!}\int \prod_{i=1}^{N} \mathrm{d}^3\vec{x}_i\, \mathrm{d}^3\vec{p}_i \exp\left[-\beta H\right] \\ &= \frac{1}{h^{3N}N!}\int \prod_{i=1}^{N} \mathrm{d}^3\vec{p}_i \exp\left[-\beta H\right] \underbrace{\int \prod_{i=1}^{N} \mathrm{d}^3\vec{x}_i}_{V} \\ &= \frac{V}{h^{3N}N!}\int \prod_{i=1}^{N} \mathrm{d}^3\vec{p}_i \exp\left[-\beta H\right] \end{aligned}$$

Substituiere $E = \frac{1}{2m}\sum_i \vec{p}_i^2$

$$\begin{aligned} &= \frac{V}{h^{3N}N!}\int_0^\infty \mathrm{d}E \prod_{i=1}^{N} \mathrm{d}^3\vec{p}_i \exp\left[-\beta E\right] \\ &= \frac{V}{h^{3N}N!}\int_0^\infty \mathrm{d}E \exp\left[-\beta E\right] \int_{S(\sqrt{2mE})} \prod_{i=1}^{N} \mathrm{d}^3\vec{p}_i \\ &= \frac{V}{h^{3N}N!}(2\pi m)^{\frac{3N}{2}} \int_0^\infty \mathrm{d}E \exp\left[-\beta E\right] \frac{E^{\frac{3N}{2}-1}}{\Gamma(\frac{3N}{2})} \\ &= \int_0^\infty \mathrm{d}E G(E) \exp\left[-\beta E\right]; \quad G(E) = \frac{V}{h^{3N}N!}(2\pi m)^{\frac{3N}{2}} \frac{E^{\frac{3N}{2}-1}}{\Gamma(\frac{3N}{2})} \\ &= \frac{V^n (2\pi m)^{\frac{3N}{2}}}{h^{3N}N!\Gamma(\frac{3N}{2})} \int_0^\infty \mathrm{d}E\; E^{\frac{3N}{2}-1} \exp\left[-\beta E\right] \\ &= \frac{V^n}{N!}\sqrt{\frac{2\pi m}{\beta h^2}}^{3N} \end{aligned}$$

b)

$G(E)$ ist folgendermaßen eine Funktion für die Anzahl der Zustände im entsprechenden Energieintervall $[E, E + \mathrm{d}E]$

c)

Es folgt mit $m = 85.47 \cdot 1.661 \cdot 10^{-27}\,\mathrm{kg}$:

$$\sqrt{\frac{\beta h^2}{2\pi m}} \approx 1.9 \cdot 10^{-6} m$$

Aufgabe 28

a)

$$\begin{aligned}
Z_\beta &= \frac{1}{h^{3N}N!}\int\prod_{i=1}^{N}\mathrm{d}^3\vec{x}_i\,\mathrm{d}^3\vec{p}_i\exp\left[-\frac{\beta}{2m}\sum_i p_i^2\right]\\
&= \frac{1}{N!}\prod_{i=1}^{N}\frac{1}{h^3}\int_V \mathrm{d}^3x_i\int\mathrm{d}^3p_i\exp\left[-\frac{\beta}{2m}\vec{p}_i^2\right] := \frac{Z^N}{N!}\\
Z &= \frac{1}{h^3}\int_V \mathrm{d}^3x_i\int\mathrm{d}^3p\exp\left[-\frac{\beta}{2m}\vec{p}^2\right] = V\cdot\sqrt{\frac{2\pi m}{\beta h^2}}^3\\
Z_\beta &= \frac{V^N}{N!}\sqrt{\frac{2\pi m}{\beta h^2}}^{3N}
\end{aligned}$$

Ergebnisse sind identisch. *alternativ:*

$$\begin{aligned}
z &= \int_0^\infty \mathrm{d}E\exp\left[-\beta E\right]\cdot\frac{1}{h^3}\int_v\mathrm{d}^3x\int_{S(E)}\mathrm{d}^3p\\
&= \frac{1}{h^3}\int_V\mathrm{d}^3x\int_{S(E)}\mathrm{d}^3p = \frac{V}{h^3}\sqrt{2m\pi}^3E^{\frac{3}{2}-1}\frac{2}{\sqrt{\pi}} = V\frac{4\pi\sqrt{2m}}{h^3}E^{\frac{3}{2}-1}
\end{aligned}$$

b)

$$\begin{aligned}
< H > &= \frac{1}{Z_\beta}\frac{1}{h^{3N}N!}\int\prod_{i=1}^{N}\mathrm{d}^3\vec{x}_i\,\mathrm{d}^3\vec{p}_iH(x)\exp\left[-\frac{\beta}{2m}\sum_i p_i^2\right]\\
&= \frac{1}{Z^N}\prod_{i=1}^{N}\int\mathrm{d}E_iE_ig(E_i)\exp(-\beta E_i)\\
&= \left(\frac{\int\mathrm{d}Eg(E)E\exp(-\beta E)}{\int\mathrm{d}Eg(E)\exp(-\beta E)}\right)^N
\end{aligned}$$

Für ein einzelnes Atom folgt für die mittlere Energie:

$$U = < H > = \left(\frac{\int\mathrm{d}EE^{\frac{3}{2}}\exp(-\beta E)}{\int\mathrm{d}EE^{\frac{3}{2}-1}\exp(-\beta E)}\right) = \frac{1}{\beta}\frac{\Gamma(\frac{3}{2}+1)}{\Gamma(\frac{3}{2})} = \frac{3}{2}kT$$

c)

$$U = \frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{1}{Z}V\sqrt{\frac{2m\pi}{h^2}}^3\frac{\partial}{\partial\beta}\beta^{-3/2} = \frac{3}{2}\frac{1}{\beta} = \frac{3}{2}kT$$

Aufgabe 29

a)

$$z = \int_0^\infty \mathrm{d}E g(E) \exp(-\beta E) = \sum_{i=0}^{\infty} \int_0^\infty \mathrm{d}E \delta(E - nE_0) \exp(-\beta E) = \sum_{i=0}^{\infty} \exp(-\beta n E_0)$$

$$\Rightarrow Z = \frac{1}{N!} z^N = \frac{1}{N!} \left(\sum_{i=0}^{\infty} \exp(-\beta n E_0) \right)^N = \frac{1}{N!} \left(\frac{\exp(\beta E_0)}{\exp(\beta E_0) - 1} \right)^N$$