Übungen zur Theoretischen Mechanik
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Blatt 02 - Lösungen

Stilianos Louca

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(2) Massenpunkt auf Kreis

In Kugelkoordinaten gilt im allgemeinen:

$$\vec{r} = r \cdot \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix}$$

$$\vec{g}_r = \frac{\partial\vec{r}}{\partial r} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix},\ \vec{g}_\theta = \frac{\partial\vec{r}}{\partial\theta} = r \cdot \begin{bmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{bmatrix},\ \vec{g}_\phi = \frac{\partial\vec{r}}{\partial\varphi} = r \cdot \begin{bmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{bmatrix}$$

$$\lambda_r := |\vec{g}_r| = 1,\ \lambda_\theta := |\vec{g}_\theta| = r,\ \lambda_\varphi := |\vec{g}_\varphi| = r\sin\theta$$

$$\vec{e}_r := \frac{\vec{g}_r}{\lambda_r},\ \vec{e}_\theta := \frac{\vec{g}_\theta}{\lambda_\theta},\ \vec{e}_\varphi := \frac{\vec{g}_\varphi}{\lambda_\varphi}$$

$$\dot{\vec{e}}_r = \begin{bmatrix} \dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi \\ \dot\theta\cos\theta\sin\phi + \dot\varphi\sin\theta\cos\phi \\ -\dot\theta\sin\theta \end{bmatrix} = \dot\theta \cdot \vec{e}_\theta + \dot\varphi\sin\theta \cdot \vec{e}_\varphi$$

$$\dot{\vec{e}}_\theta = \begin{bmatrix} -\dot\theta\sin\theta\cos\phi - \dot\varphi\cos\theta\sin\varphi \\ -\dot\theta\sin\theta\sin\varphi + \dot\varphi\cos\theta\cos\phi \\ -\dot\theta\cos\theta \end{bmatrix} = -\dot\theta \cdot \vec{e}_r + \dot\varphi\cos\theta \cdot \vec{e}_\varphi$$

$$\dot{\vec{e}}_\varphi = \begin{bmatrix} -\dot\varphi\cos\phi \\ -\dot\varphi\sin\varphi \\ 0 \end{bmatrix} = -\dot\varphi\sin\theta \cdot \vec{e}_r - \dot\varphi\cos\theta \cdot \vec{e}_\theta$$

$$\dot{\vec{r}} = \dot r \cdot \vec{g}_r + \dot\theta \cdot \vec{g}_\theta + \dot\varphi \cdot \vec{g}_\varphi = \dot r \cdot \vec{e}_r + r\dot\theta \cdot \vec{e}_\theta + r\dot\varphi\sin\theta \cdot \vec{e}_\varphi$$

$$\ddot{\vec{r}} = \left(\ddot r - r\dot\theta^2 - r\dot\varphi^2\sin^2\theta\right) \cdot \vec{e}_r + \left(r\ddot\theta + 2\dot r\dot\theta - r\dot\varphi^2\sin\theta\cos\theta\right) \cdot \vec{e}_\theta + \left(r\ddot\varphi\sin\theta + 2\dot r\dot\varphi\sin\theta + 2r\dot\theta\dot\varphi\cos\theta\right) \cdot \vec{e}_\varphi$$

- **Kugelkoordinaten:**

$$r = R_0 : const \;\Rightarrow\; \dot{r} = 0,\; \dot{\phi} = \omega : const \;\Rightarrow\; \ddot{\varphi} = 0$$

$$\Rightarrow\; \dot{\vec{r}} = R_0\dot{\theta} \cdot \vec{e}_\theta + \omega R_0 \sin\theta \cdot \vec{e}_\varphi \;\Rightarrow\; |\dot{\vec{r}}| = v_0 = R_0\sqrt{\dot{\theta}^2 + \omega^2 \sin^2\theta} : const \;\Rightarrow\; \dot{\theta} = \sqrt{\frac{v_0^2}{R_0^2} - \omega^2 \sin^2\theta} = \omega\cos\theta \quad (da\ v_0 = \omega R_0)$$

$$\ddot{\theta} = \frac{d\left(\omega\cos\theta\right)}{dt} = -\omega\dot{\theta}\sin\theta = -\omega^2\cos\theta\sin\theta$$

$$\Rightarrow\; \dot{\vec{r}} = v_0\cos\theta \cdot \vec{e}_\theta + v_0\sin\theta \cdot \vec{e}_\varphi$$

$$\ddot{\vec{r}} = -\left(\omega^2 R_0\cos^2\theta + \omega^2 R_0 \sin^2\theta\right) \cdot \vec{e}_r - \left(\omega^2 R_0 \cos\theta\sin\theta + \omega^2 R_0\sin\theta\cos\theta\right) \cdot \vec{e}_\theta + 2\omega^2 R_0\cos^2\theta \cdot \vec{e}_\varphi$$

$$= \omega^2 R_0 \cdot \left(-\vec{e}_r - 2\cos\theta\sin\theta \cdot \vec{e}_\theta + 2\cos^2\theta \cdot \vec{e}_\varphi\right)$$

Normalkoordinaten:
Sind $\vec{T}$ der Bahn-Tangenteneinheitsvektor und $\vec{N}$ der Bahn-Normaleinheitsvektor, so gilt

$$\dot{\vec{r}} = v_0 \cdot \vec{T} \;\Rightarrow\; \vec{T} = \cos\theta \cdot \vec{e}_\theta + \sin\theta \cdot \vec{e}_\varphi = \begin{pmatrix} \cos^2\theta\cos\varphi - \sin\theta\sin\varphi \\ \cos^2\theta\sin\varphi + \sin\theta\cos\varphi \\ -\cos\theta\sin\theta \end{pmatrix}$$

$$\ddot{\vec{r}} = \dot{v_0} \cdot \vec{T} + \frac{v_0^2}{R} \cdot \vec{N} = \frac{v_0^2}{R} \cdot \vec{N} \;\Rightarrow\; \frac{v_0^2}{R} = \left|\ddot{\vec{r}}\right| \;\Rightarrow\; R = \frac{v_0^2}{\left|\ddot{\vec{r}}\right|} = \frac{R_0}{\sqrt{1+4\cos^2\theta}}$$

$$\Rightarrow\; \vec{N} = \frac{-\vec{e}_r - 2\cos\theta\sin\theta \cdot \vec{e}_\theta + 2\cos^2\theta \cdot \vec{e}_\varphi}{\sqrt{1+4\cos^2\theta}} = \frac{1}{\sqrt{1+4\cos^2\theta}} \cdot \begin{bmatrix} -\sin\theta\cos\varphi(1+2\cos^2\theta) - 2\cos^2\theta\sin\varphi \\ -\sin\theta\sin\varphi(1+2\cos^2\theta) + 2\cos^2\theta\cos\varphi \\ \cos\theta(2\sin^2\theta - 1) \end{bmatrix}$$

- $$\dot{\theta} = \omega\cos\theta \;\Rightarrow\; \int_{\theta_0}^{\theta}\frac{du}{\cos u} = \int_{t_0}^{t}\omega d\lambda \;\Rightarrow\; \ln\left|\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right| = \omega(t-t_0) + C,\; C\in\mathbb{R}$$

$$\Rightarrow\; \theta(t) = 2\arctan\left(Ae^{\omega(t-t_0)}\right) - \frac{\pi}{2},\; A\in\mathbb{R}^+$$

$$R(t) = \frac{R_0}{\sqrt{1+4\cos^2\theta(t)}} = \frac{R_0}{\sqrt{1+\frac{4A^2e^{2\omega(t-t_0)}}{\left(1+A^2e^{2\omega(t-t_0)}\right)^2}}}$$

$$F\ddot{u}r\ (\theta_0, t_0) \overset{!}{=} (0,0) \;\rightarrow\; A = 1 \;\rightarrow\; \theta(t) = 2\arctan\left(e^{\omega t}\right) - \frac{\pi}{2},\; R(t) = \frac{R_0}{\sqrt{1+\frac{4e^{2\omega t}}{(1+e^{2\omega t})^2}}}$$

$$\lim_{t\to\infty}\theta = \frac{\pi}{2},\; \lim_{t\to\infty} R = \lim_{\theta\to\pi/2}\frac{R_0}{\sqrt{1+4\cos^2\theta}} = R_0$$

(3) Harmonische Schwingungen

$$f(t) = \alpha_0 |t| = \left\{ \begin{array}{ll} \alpha_0 t & : t \geq 0 \\ -\alpha_0 t & : t < 0 \end{array} \right\} = \sum_{n=-\infty}^{\infty} A_n e^{in\omega t}, \ \omega := \frac{2\pi}{T}$$

$$Für\ n \neq 0 : A_n = \frac{1}{T} \cdot \int_{-T/2}^{T/2} f(t) e^{-in\omega t} \cdot dt = \frac{\alpha_0}{T} \cdot \left[-\int_{-T/2}^{0} t e^{-in\omega t} \cdot dt + \int_{0}^{T/2} t e^{-in\omega t} \cdot dt \right]$$

$$= \frac{\alpha_0}{T} \cdot \left[\int_{0}^{-T/2} t e^{-in\omega t} \cdot dt + \int_{0}^{T/2} t e^{-in\omega t} \cdot dt \right]$$

$$= \frac{\alpha_0}{T} \cdot \left[e^{-in\omega t} \cdot \left(\frac{it}{\omega n} + \frac{1}{n^2 \omega^2} \right) \right]_0^{-\pi/\omega} + \frac{\alpha_0}{T} \cdot \left[e^{-in\omega t} \cdot \left(\frac{it}{\omega n} + \frac{1}{n^2 \omega^2} \right) \right]_0^{\pi/\omega}$$

$$= \frac{2\alpha_0}{T} \cdot \left[\frac{(-1)^n}{n^2 \omega^2} - \frac{1}{n^2 \omega^2} \right] = \left\{ \begin{array}{ll} 0 & : n\ gerade \\ \frac{-\alpha_0 T}{n^2 \pi^2} & : n\ ungerade \end{array} \right.$$

$$A_0 = \frac{\alpha_0}{T} \cdot \left[\int_{0}^{-T/2} t \cdot dt + \int_{0}^{T/2} t \cdot dt \right] = \frac{\alpha_0}{T} \cdot \left(\left[\frac{t^2}{2} \right]_0^{-T/2} + \left[\frac{t^2}{2} \right]_0^{T/2} \right) = \frac{\alpha_0 T}{4}$$

Es gilt: $A_n = A_{-n} \ \forall \ n \in \mathbb{N}$

Demzufolge:

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega t} = A_0 + \sum_{n=1}^{\infty} \left(A_n e^{in\omega t} + A_{-n} e^{-in\omega t} \right) = A_0 + \sum_{n=1}^{\infty} 2A_n \cos(n\omega t)$$

$$= \frac{\alpha_0 T}{4} - \frac{2\alpha_0 T}{\pi^2} \cdot \sum_{n=1}^{\infty} \frac{\cos\left((2n-1)\omega t\right)}{(2n-1)^2}$$