Relativistische Physik

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Übungsserie 08 - Lösungen

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Aufgabe 15

Beginnend mit der Parameter-Umrechnungsvorschrift $\mu = \mu(\lambda)$, das heißt

$$\frac{dx^m}{d\lambda} = \frac{dx^m}{d\mu}\frac{d\mu}{d\lambda}$$

$$\frac{d^2x^m}{d\lambda^2} = \frac{d\mu}{d\lambda}\frac{d}{d\lambda}\frac{dx^m}{d\mu} + \frac{dx^m}{d\mu}\frac{d^2\mu}{d\lambda^2} = \frac{d^2x^m}{d\mu^2}\cdot\left[\frac{d\mu}{d\lambda}\right]^2 + \frac{dx^m}{d\mu}\cdot\frac{d^2\mu}{d\lambda^2}$$

und der Geodätengleichung schreiben wir

$$0 = \frac{d^2x^m}{d\lambda^2} + \Gamma^m_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = \frac{d^2x^m}{d\mu^2}\cdot\left[\frac{d\mu}{d\lambda}\right]^2 + \frac{dx^m}{d\mu}\cdot\frac{d^2\mu}{d\lambda^2} + \Gamma^m_{ij}\frac{dx^i}{d\mu}\frac{dx^j}{d\mu}\left[\frac{\partial\mu}{\partial\lambda}\right]^2$$

$$= \left[\frac{d\mu}{d\lambda}\right]^2\cdot\left[\frac{d^2x^m}{d\mu^2} + \frac{dx^m}{d\mu}\frac{d^2\mu}{d\lambda^2}\left(\frac{d\mu}{d\lambda}\right)^{-2} + \Gamma^m_{ij}\frac{dx^i}{d\mu}\frac{dx^j}{d\mu}\right]$$

$$= \left[\frac{d\mu}{d\lambda}\right]^2\cdot\left[\frac{d^2x^m}{d\mu^2} - \frac{dx^m}{d\mu}\frac{d}{d\lambda}\left(\frac{d\mu}{d\lambda}\right)^{-1} + \Gamma^m_{ij}\frac{dx^i}{d\mu}\frac{dx^j}{d\mu}\right]$$

und erhalten

$$\boxed{\frac{d^2x^m}{d\mu^2} - \frac{dx^m}{d\mu}\frac{d}{d\lambda}\left(\frac{d\mu}{d\lambda}\right)^{-1} + \Gamma^m_{ij}\frac{dx^i}{d\mu}\frac{dx^j}{d\mu} = 0} \tag{1}$$

Euler-Lagrange Gleichungen

Alternativ könnte man nach den *nativen* Bewegungsgleichungen bzgl. des Parameters μ fragen. Beginnend mit den Euler-Lagrange Gleichungen

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{u}^k} - \frac{\partial\mathcal{L}}{\partial u^k} = 0 \quad , \quad \mathcal{L}(u,\dot{u}) := \sqrt{g_{ij}\dot{u}^i\dot{u}^j}$$

und

$$\frac{\partial\mathcal{L}}{\partial\dot{u}^k} = \frac{1}{2\mathcal{L}}\cdot g_{ij}\left(\delta^i_k\dot{u}^j + \delta^j_k\dot{u}^i\right) = \frac{g_{ik}\dot{u}^i}{\mathcal{L}}$$

$$\frac{\partial\mathcal{L}}{\partial u^k} = \frac{\dot{u}^i\dot{u}^j}{2\mathcal{L}}\cdot\frac{\partial g_{ij}}{\partial u^k}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{u}^k} = -\frac{g_{ik}\dot{u}^i}{\mathcal{L}^2}\left[\frac{\partial\mathcal{L}}{\partial\dot{u}^r}\ddot{u}^r + \frac{\partial\mathcal{L}}{\partial u^r}\dot{u}^r\right] + \frac{\dot{u}^i}{\mathcal{L}}\frac{\partial g_{ik}}{\partial u^r}\dot{u}^r + \frac{g_{ik}}{\mathcal{L}}\ddot{u}^i = -\frac{g_{ik}\dot{u}^i}{\mathcal{L}^2}\left[\frac{1}{\mathcal{L}}\underbrace{g_{jr}\dot{u}^j\ddot{u}^r}_{0} + \frac{\dot{u}^l\dot{u}^j\dot{u}^r}{2\mathcal{L}}\frac{\partial g_{lj}}{\partial u^r}\right] + \frac{\dot{u}^i\dot{u}^j}{\mathcal{L}}\frac{\partial g_{ik}}{\partial u^j} + \frac{g_{ik}}{\mathcal{L}}\ddot{u}^i$$

schreiben wir

$$0 = g_{ik}\ddot{u}^i - \frac{g_{ik}}{2\mathcal{L}^2}\dot{u}^i\dot{u}^l\dot{u}^j\dot{u}^r\frac{\partial g_{lj}}{\partial u^r} + \dot{u}^i\dot{u}^j\frac{\partial g_{ik}}{\partial u^j} - \frac{\dot{u}^i\dot{u}^j}{2}\frac{\partial g_{ij}}{\partial u^k}$$

$$= g_{ik}\ddot{u}^i - \frac{g_{ik}}{2\mathcal{L}^2}\dot{u}^i\dot{u}^l\dot{u}^j\dot{u}^r\frac{\partial g_{lj}}{\partial u^r} + \frac{\dot{u}^i\dot{u}^j}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i}\right) - \frac{\dot{u}^i\dot{u}^j}{2}\frac{\partial g_{ij}}{\partial u^k}$$

$$\overset{\times g^{ks}}{\Longleftrightarrow} \quad 0 = \underbrace{g_{ik}g^{ks}}_{\delta^s_k}\ddot{u}^i - \frac{g_{ik}g^{ks}}{2\mathcal{L}^2}\dot{u}^i\dot{u}^l\dot{u}^j\dot{u}^r\frac{\partial g_{lj}}{\partial u^r} + \dot{u}^i\,\dot{u}^j\underbrace{\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right)g^{ks}}_{\Gamma^s_{ij}}$$

$$= \ddot{u}^s - \frac{1}{2\mathcal{L}^2}\dot{u}^s\dot{u}^l\dot{u}^j\dot{u}^r\frac{\partial g_{lj}}{\partial u^r} + \Gamma^s_{ij}\dot{u}^i\dot{u}^j$$

und erhalten die allgemeineren Geodätengleichungen

$$\boxed{\ddot{u}^s = \frac{\dot{u}^s\dot{u}^l\dot{u}^j\dot{u}^r}{2g_{qp}\dot{u}^q\dot{u}^p}\frac{\partial g_{lj}}{\partial u^r} - \Gamma^s_{ij}\dot{u}^i\dot{u}^j} \tag{2}$$

Korollar 01

Für skalare S_{ij}, A^{ij} mit $S_{ij} = S_{ji}$ (symmetrisch) und $A^{ij} = -A^{ji}$ (antisymmetrisch) ist

$$S_{ij}A^{ij} = 0 \tag{3}$$

Beweis:

$$2S_{ij}A^{ij} = S_{ij}A^{ij} + S_{ij}A^{ij} \overset{i\leftrightarrow j}{=} S_{ij}A^{ij} + \underbrace{S_{ji}}_{S_{ij}}\underbrace{A^{ji}}_{-A^{ij}} = S_{ij}A^{ij} - S_{ij}A^{ij} = 0$$

□

Aufgabe 16

a) Beginnend mit den inhomogenen Maxwellgleichungen

$$\nabla_\mu F^{\nu\mu} = \mu_0 j^\nu$$

in der kovarianten Formulierung schreiben wir

$$\mu_0\nabla_\nu j^\nu = \nabla_\nu\nabla_\mu F^{\nu\mu} = \nabla_\nu\left[\partial_\mu F^{\nu\mu} + \underbrace{\Gamma^\nu_{d\mu}F^{d\mu}}_{\substack{0\\ \text{nach (3)}}} + \Gamma^\mu_{d\mu}F^{\nu d}\right]$$

$$= \underbrace{\partial_\nu\partial_\mu F^{\nu\mu}}_{\substack{0\\ \text{nach (3)}}} + \Gamma^\nu_{d\nu}\partial_\mu F^{d\mu} + \partial_\nu\left(\Gamma^\mu_{d\mu}F^{\nu d}\right) + \underbrace{\Gamma^\nu_{c\nu}\Gamma^\mu_{d\mu}F^{cd}}_{\substack{0\\ \text{nach (3)}}}$$

$$= \Gamma^\nu_{d\nu}\partial_\mu F^{d\mu} + \underbrace{\Gamma^\mu_{d\mu}\partial_\nu F^{\nu d}}_{\Gamma^\nu_{d\nu}\partial_\mu F^{\mu d}} + F^{\nu d}\partial_\nu\Gamma^\mu_{d\mu} = \cancel{\Gamma^\nu_{d\nu}\partial_\mu F^{d\mu}} - \cancel{\Gamma^\nu_{d\nu}\partial_\mu F^{d\mu}} + F^{\nu d}\partial_\nu\Gamma^\mu_{d\mu}$$

und erhalten die Kontinuitätsgleichung

$$\nabla_\nu j^\nu = \frac{1}{\mu_0}F^{\nu d}\partial_\nu\Gamma^\mu_{d\mu}$$

Mit

$$\Gamma^\mu_{d\mu} = \frac{\partial \ln\sqrt{-g}}{\partial x^d}$$

folgt dann nach Korollar 01

$$\boxed{\nabla_\nu j^\nu = 0} \tag{4}$$

b) Beginnend mit

$$F^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu \tag{5}$$

schreiben wir

$$-\mu_0 j^\nu = \nabla_\mu F^{\mu\nu} = \underbrace{\nabla_\mu \nabla^\mu A^\nu}_{\Box A^\nu} - \nabla_\mu \nabla^\nu A^\mu = \Box A^\nu - \nabla_\mu \nabla^\nu A^\mu + \nabla^\nu \overbrace{\nabla_\mu A^\mu}^{0}$$

$$= \Box A^\nu + g^{\nu n} \left[\nabla_n \nabla_\mu A^\mu - \nabla_\mu \nabla_n A^\mu\right] = \Box A^\nu + g^{\nu n} \left[\nabla_n \left(\partial_\mu A^\mu + \Gamma^\mu_{\mu d} A^d\right) - \nabla_\mu \left(\partial_n A^\mu + \Gamma^\mu_{nd} A^d\right)\right]$$

$$= \Box A^\nu + g^{\nu n} \left[\cancel{\partial_n \partial_\mu A^\mu} + \partial_n \left(\Gamma^\mu_{\mu d} A^d\right) - \left(\cancel{\partial_\mu \partial_n A^\mu} + \Gamma^\mu_{\mu c} \partial_n A^c - \Gamma^c_{\mu n} \partial_c A^\mu + \partial_\mu \left(\Gamma^\mu_{nd} A^d\right) + \Gamma^\mu_{\mu c} \Gamma^c_{nd} A^d - \Gamma^c_{\mu n} \Gamma^\mu_{cd} A^d\right)\right]$$

$$= \Box A^\nu + g^{\nu n} \left[\cancel{\Gamma^\mu_{\mu d} \partial_n A^d} + A^d \partial_n \Gamma^\mu_{\mu d} - \cancel{\Gamma^\mu_{\mu c} \partial_n A^c} + \cancel{\Gamma^c_{\mu n} \partial_c A^\mu} - \cancel{\Gamma^\mu_{nd} \partial_\mu A^d} - A^d \partial_\mu \Gamma^\mu_{nd} - \Gamma^\mu_{\mu c} \Gamma^c_{nd} A^d + \Gamma^c_{\mu n} \Gamma^\mu_{cd} A^d\right]$$

$$= \Box A^\nu + g^{n\nu} \underbrace{\left[\partial_n \Gamma^\mu_{\mu d} + \Gamma^c_{\mu n} \Gamma^\mu_{cd} - \partial_\mu \Gamma^\mu_{nd} - \Gamma^\mu_{\mu c} \Gamma^c_{nd}\right]}_{-R_{dn}} A^d$$

und erhalten

$$\boxed{\Box A^\nu = -\mu_0 j^\nu + R^\nu_d A^d} \tag{6}$$

mit dem Ricci-Tensor R_{dn}.