Relativistische Physik

FSU Jena - WS 2008/2009

Übungsserie 07 - Lösungen

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Aufgabe 13

Es sei T^a ein (1,0)-Tensor auf der n-dimensionalen semi-Riemannschen Mannigfaltigkeit (M,g), darauf die Karte ψ in den Koordinaten x^i. Zu zeigen wäre: die Definition von ∇T als (1,1)-Tensor durch

$$(\nabla T)^a_m := \frac{\partial T^a}{\partial x^m} + \Gamma^a_{mn} T^n \tag{1}$$

ist **Kartenunabhängig**[1], das heißt die obige Definition ergibt in jeweils zwei beliebigen Karten den gleichen Tensor.

Beweis: Es sei $\tilde\psi$ eine weitere Karte in den Koordinaten $\tilde x^i$ und ∇T definiert in dieser Karte nach Gleichung 1. Es seien $S^b_m, \tilde S^\beta_\mu$ die Komponenten dieses Tensors jeweils in den Karten ψ und $\tilde\psi$. Dann gilt:

$$S^b_m = \underbrace{\left[\frac{\partial \tilde T^\beta}{\partial \tilde x^\mu} + \tilde\Gamma^\beta_{\mu\nu}\tilde T^\nu\right]}_{\substack{\tilde S^\beta_\mu \\ \text{per Konstruktion}}} \frac{\partial \tilde x^\mu}{\partial x^m}\frac{\partial x^b}{\partial \tilde x^\beta} = \frac{\partial \tilde T^\beta}{\partial x^m}\frac{\partial x^b}{\partial \tilde x^\beta} + \tilde\Gamma^\beta_{\mu\nu}\frac{\partial \tilde x^\mu}{\partial x^m}\frac{\partial x^b}{\partial \tilde x^\beta}\overbrace{\frac{\partial \tilde x^\nu}{\partial x^n}T^n}^{\tilde T^\nu}$$

$$= \frac{\partial}{\partial x^m}\left(\underbrace{\tilde T^\beta \frac{\partial x^b}{\partial \tilde x^\beta}}_{T^b}\right) - \tilde T^\beta \frac{\partial^2 x^b}{\partial x^m \partial \tilde x^\beta} + \tilde\Gamma^\beta_{\mu\nu}\frac{\partial\tilde x^\mu}{\partial x^m}\frac{\partial x^b}{\partial\tilde x^\beta}\frac{\partial \tilde x^\nu}{\partial x^n}T^n = \frac{\partial T^b}{\partial x^m} - T^n\frac{\partial \tilde x^\beta}{\partial x^n}\frac{\partial^2 x^b}{\partial x^m\partial\tilde x^\beta} + \tilde\Gamma^\beta_{\mu\nu}\frac{\partial\tilde x^\mu}{\partial x^m}\frac{\partial x^b}{\partial\tilde x^\beta}\frac{\partial \tilde x^\nu}{\partial x^n}T^n$$

$$\overset{\text{Lemma}}{=} \frac{\partial T^b}{\partial x^m} - T^n \frac{\partial\tilde x^\beta}{\partial x^n}\frac{\partial^2 \tilde x^b}{\partial\tilde x^\mu\partial\tilde x^\beta}\frac{\partial\tilde x^\mu}{\partial x^m} + \Gamma^j_{rs}\underbrace{\frac{\partial x^r}{\partial\tilde x^\mu}\frac{\partial\tilde x^\mu}{\partial x^m}}_{\delta^r_m}\underbrace{\frac{\partial x^s}{\partial\tilde x^\nu}\frac{\partial\tilde x^\nu}{\partial x^n}}_{\delta^s_n}\underbrace{\frac{\partial\tilde x^\beta}{\partial x^j}\frac{\partial x^b}{\partial\tilde x^\beta}}_{\delta^b_j} T_n + \frac{\partial^2 x^s}{\partial\tilde x^\mu\partial\tilde x^\nu}\underbrace{\frac{\partial\tilde x^\beta}{\partial x^s}\frac{\partial x^b}{\partial\tilde x^\beta}}_{\delta^b_s}\frac{\partial\tilde x^\mu}{\partial x^m}\frac{\partial\tilde x^\nu}{\partial x^n}T^n$$

$$= \frac{\partial T^b}{\partial x^m} \cancel{- T^n\frac{\partial\tilde x^\beta}{\partial x^n}\frac{\partial^2\tilde x^b}{\partial\tilde x^\mu\partial\tilde x^\beta}\frac{\partial\tilde x^\mu}{\partial x^m}} + \Gamma^b_{mn}T^n \cancel{+ \frac{\partial^2 x^b}{\partial\tilde x^\mu\partial\tilde x^\nu}\frac{\partial\tilde x^\mu}{\partial x^m}\frac{\partial\tilde x^\nu}{\partial x^n}T^n}$$

$$= \frac{\partial T^b}{\partial x^m} + \Gamma^b_{mn}T^n$$

□

Aufgabe 14

Beginnend mit der allgemeinen Formel

$$\nabla_s T^{i_1,..,i_q}_{j_1,..,j_p} := (\nabla_s T)^{i_1,..,i_q}_{j_1,..,j_p} = \frac{\partial}{\partial s}T^{i_1,..,i_q}_{j_1,..,j_p} + \sum_{k=1}^q \Gamma^{i_k}_{ds} T^{i_1,..,i_{k-1},d,i_{k+1},..,i_q}_{j_1,..,j_p} - \sum_{k=1}^p \Gamma^d_{j_k s} T^{i_1,..,i_q}_{j_1,..,j_{k-1},d,j_{k+1},..,j_p}$$

[1] Solange festgelegt ist auf welche Koordinaten man sich bezieht, kann jedes Zahlen-Tubel $T^{j_1,..,j_q}_{i_1,..,i_p}$ mit einem Tensor identifiziert werden! Die Frage hier ist: Ist dann die Darstellung von T in anderen Karten gleich?

bzw. dem Spezialfall

$$\nabla_q T_{mn} = \frac{\partial T_{mn}}{\partial x^q} - \Gamma^d_{mq} T_{dn} - \Gamma^d_{nq} T_{md}$$

schreiben wir

$$\nabla_q \nabla_s T_m = \nabla_q \left[\frac{\partial T_n}{\partial x^s} - \Gamma^n_{sm} T_n \right] \overset{\text{Linearität}}{=} \nabla_q \left(\frac{\partial T_m}{\partial x^s} \right) - \nabla_q \left(\Gamma^n_{sm} T_n \right)$$

$$= \frac{\partial}{\partial x^q} \left(\frac{\partial T_m}{\partial x^s} \right) - \Gamma^d_{qm} \frac{\partial T_d}{\partial x^s} - \Gamma^d_{qs} \frac{\partial T_n}{\partial x^d} - \frac{\partial}{\partial x^q} \left(\Gamma^n_{sm} T_n \right) + \Gamma^d_{qs} \Gamma^n_{dm} T_n + \Gamma^d_{qm} \Gamma^n_{sd} T_n$$

$$= \frac{\partial^2 T_m}{\partial x^q \partial x^s} - \Gamma^d_{qm} \frac{\partial T_d}{\partial x^s} - \Gamma^d_{qs} \frac{\partial T_n}{\partial x^d} - \frac{\partial \Gamma^n_{sm}}{\partial x^q} T_n - \Gamma^n_{sm} \frac{\partial T_n}{\partial x^q} + \Gamma^d_{qs} \Gamma^n_{dm} T_n + \Gamma^d_{qm} \Gamma^n_{sd} T_n$$

$$\nabla_s \nabla_q T_m = \frac{\partial^2 T_m}{\partial x^s \partial x^q} - \Gamma^d_{sm} \frac{\partial T_d}{\partial x^q} - \Gamma^d_{sq} \frac{\partial T_n}{\partial x^d} - \frac{\partial \Gamma^n_{qm}}{\partial x^s} T_n - \Gamma^n_{qm} \frac{\partial T_n}{\partial x^s} + \Gamma^d_{sq} \Gamma^n_{dm} T_n + \Gamma^d_{sm} \Gamma^n_{qd} T_n$$

so dass folgt

$$\nabla_q \nabla_s T_m - \nabla_s \nabla_q T_m = \cancel{\frac{\partial^2 T_m}{\partial x^q \partial x^s}} - \cancel{\Gamma^d_{qm} \frac{\partial T_d}{\partial x^s}} - \cancel{\Gamma^d_{qs} \frac{\partial T_n}{\partial x^d}} - \frac{\partial \Gamma^n_{sm}}{\partial x^q} T_n - \cancel{\Gamma^n_{sm} \frac{\partial T_n}{\partial x^q}} + \cancel{\Gamma^d_{qs} \Gamma^n_{dm} T_n} + \Gamma^d_{qm} \Gamma^n_{sd} T_n$$

$$- \cancel{\frac{\partial^2 T_m}{\partial x^s \partial x^q}} + \cancel{\Gamma^d_{sm} \frac{\partial T_d}{\partial x^q}} + \cancel{\Gamma^d_{sq} \frac{\partial T_n}{\partial x^d}} + \frac{\partial \Gamma^n_{qm}}{\partial x^s} T_n + \cancel{\Gamma^n_{qm} \frac{\partial T_n}{\partial x^s}} - \cancel{\Gamma^d_{sq} \Gamma^n_{dm} T_n} - \Gamma^d_{sm} \Gamma^n_{qd} T_n$$

$$= \underbrace{\left[\Gamma^d_{qm} \Gamma^n_{sd} - \Gamma^d_{sm} \Gamma^n_{qd} - \frac{\partial \Gamma^n_{sm}}{\partial x^q} + \frac{\partial \Gamma^n_{qm}}{\partial x^s} \right]}_{R^n_{msq}} T_n$$

Lemma über die Christoffel-Symbole

Es sei (M,g) eine semi-Riemannsche n-dimensionale Mannigfaltigkeit, mit Karten ψ und $\tilde\psi$ jeweils in den Koordinaten x^i und $\tilde x^i$. Dann gilt für die Christoffel-Symbole Γ^b_{mn} bzw. $\tilde\Gamma^\beta_{\mu\nu}$ in diesen Karten die Transformationsvorschrift:

$$\tilde\Gamma^\beta_{\mu\nu} = \frac{1}{2}\left[\frac{\partial\tilde g_{\mu\varkappa}}{\partial\tilde x^\nu}+\frac{\partial\tilde g_{\nu\varkappa}}{\partial\tilde x^\mu}-\frac{\partial\tilde g_{\mu\nu}}{\partial\tilde x^\varkappa}\right]\tilde g^{\varkappa\beta}$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial\tilde x^\nu}\left(g_{mk}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^k}{\partial\tilde x^\varkappa}\right)+\frac{\partial}{\tilde x^\mu}\left(g_{nk}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial x^k}{\partial\tilde x^\varkappa}\right)-\frac{\partial}{\partial\tilde x^\varkappa}\left(g_{mn}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\right)\right]\tilde g^{\varkappa\beta}$$

$$= \frac{1}{2}\left[\frac{\partial g_{mk}}{\partial\tilde x^\nu}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^k}{\partial\tilde x^\varkappa}+g_{mk}\frac{\partial^2 x^m}{\partial\tilde x^\nu\partial\tilde x^\mu}\frac{\partial x^k}{\partial\tilde x^\varkappa}+g_{mk}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial^2 x^k}{\partial\tilde x^\nu\partial\tilde x^\varkappa}\right]\tilde g^{\varkappa\beta}$$

$$+ \frac{1}{2}\left[\frac{\partial g_{nk}}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial x^k}{\partial\tilde x^\varkappa}+g_{nk}\frac{\partial^2 x^n}{\partial\tilde x^\mu\partial\tilde x^\nu}\frac{\partial x^k}{\partial\tilde x^\varkappa}+g_{nk}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial^2 x^k}{\partial\tilde x^\mu\partial\tilde x^\varkappa}\right]\tilde g^{\varkappa\beta}$$

$$- \frac{1}{2}\left[\frac{\partial g_{mn}}{\partial\tilde x^\varkappa}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}+g_{mn}\frac{\partial^2 x^m}{\partial\tilde x^\varkappa\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}+g_{mn}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial^2 x^n}{\partial\tilde x^\varkappa\partial\tilde x^\nu}\right]\tilde g^{\varkappa\beta}$$

$$= \frac{1}{2}\left[\underbrace{\frac{\partial g_{mk}}{\partial\tilde x^\nu}}_{\frac{\partial g_{mk}}{\partial x^l}\frac{\partial x^l}{\partial\tilde x^\nu}}\frac{\partial x^m}{\partial\tilde x^\mu}\underbrace{\frac{\partial x^k}{\partial\tilde x^\varkappa}\frac{\partial\tilde x^\varkappa}{\partial x^i}}_{\delta^k_i}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+g_{mk}\frac{\partial^2 x^m}{\partial\tilde x^\nu\partial\tilde x^\mu}\underbrace{\frac{\partial x^k}{\partial\tilde x^\varkappa}\frac{\partial\tilde x^\varkappa}{\partial x^i}}_{\delta^k_i}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+\cancel{g_{mk}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial^2 x^k}{\partial\tilde x^\nu\partial\tilde x^\varkappa}\tilde g^{\varkappa\beta}}\right]$$

$$+ \frac{1}{2}\left[\underbrace{\frac{\partial g_{nk}}{\partial\tilde x^\mu}}_{\frac{\partial g_{nk}}{\partial x^l}\frac{\partial x^l}{\partial\tilde x^\mu}}\frac{\partial x^n}{\partial\tilde x^\nu}\underbrace{\frac{\partial x^k}{\partial\tilde x^\varkappa}\frac{\partial\tilde x^\varkappa}{\partial x^i}}_{\delta^k_i}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+g_{nk}\frac{\partial^2 x^n}{\partial\tilde x^\mu\partial\tilde x^\nu}\underbrace{\frac{\partial x^k}{\partial\tilde x^\varkappa}\frac{\partial\tilde x^\varkappa}{\partial x^i}}_{\delta^k_i}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+\cancel{g_{nk}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial^2 x^k}{\partial\tilde x^\mu\partial\tilde x^\varkappa}\tilde g^{\varkappa\beta}}\right]$$

$$- \frac{1}{2}\left[\underbrace{\frac{\partial g_{mn}}{\partial\tilde x^\varkappa}\frac{\partial\tilde x^\varkappa}{\partial x^i}}_{\frac{\partial g_{mn}}{\partial x^i}}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+\cancel{g_{mn}\frac{\partial^2 x^m}{\partial\tilde x^\varkappa\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\tilde g^{\varkappa\beta}}+\cancel{g_{mn}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial^2 x^n}{\partial\tilde x^\varkappa\partial\tilde x^\nu}\tilde g^{\varkappa\beta}}\right]$$

$$= \frac{1}{2}\left[\frac{\partial g_{mi}}{\partial x^l}\frac{\partial x^l}{\partial\tilde x^\nu}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+\underbrace{g_{ni}g^{ij}}_{\delta^j_n}\frac{\partial^2 x^n}{\partial\tilde x^\nu\tilde x^\mu}\frac{\partial\tilde x^\beta}{\partial x^j}+\frac{\partial g_{ni}}{\partial x^l}\frac{\partial x^l}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+\underbrace{g_{ni}g^{ij}}_{\delta^j_n}\frac{\partial^2 x^n}{\partial\tilde x^\mu\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^j}-\frac{\partial g_{mn}}{\partial x^i}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}\right]$$

$$= \frac{1}{2}\left[\frac{\partial g_{mi}}{\partial x^l}\frac{\partial x^l}{\partial\tilde x^\nu}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+\frac{\partial^2 x^n}{\partial\tilde x^\mu\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^n}+\frac{\partial g_{ni}}{\partial x^l}\frac{\partial x^l}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+\frac{\partial^2 x^n}{\partial\tilde x^\mu\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^n}-\frac{\partial g_{mn}}{\partial x^i}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}\right]$$

$$= \frac{1}{2}\left[\frac{\partial g_{mi}}{\partial x^n}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}+2\frac{\partial^2 x^n}{\partial\tilde x^\mu\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^n}+\frac{\partial g_{ni}}{\partial x^m}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}-\frac{\partial g_{mn}}{\partial x^i}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^j}g^{ij}\right]$$

$$= \Gamma^j_{mn}\frac{\partial x^m}{\partial\tilde x^\mu}\frac{\partial x^n}{\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^j}+\frac{\partial^2 x^n}{\partial\tilde x^\mu\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^n}$$

Bemerke: Speziell für Riemansche Normalkoordinaten x^i ist

$$\tilde\Gamma^\beta_{\mu\nu} = \frac{\partial^2 x^n}{\partial\tilde x^\mu\partial\tilde x^\nu}\frac{\partial\tilde x^\beta}{\partial x^n}$$