Metric Tensors, Covariant Derivatives and the Geodesic Equation

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1 Generalized Riemannian manifolds

A priori, manifolds lack any coordinate-independent geometric notion such as orthogonality of vectors and distance between points. For manifolds embedded into $\mathbb{R}^n$, these notions can be naturally inherited from the larger *host* space using the restriction of the euclidean scalar product to the tangent spaces of the embedded manifold. But as in many real life applications such an embedding does not always exist for the underlying studied manifold, the need arises to define distance and volume intrinsically using only operations on the manifold its self. Riemannian manifolds generalize these notions to abstract manifolds not embedded into any higher space by means of the so called *metric tensor*, defined purely on the tangent bundle of the manifold and acting as an intermediate between its topology and geometry. Distance, geodesics, curvature, volume and differential operators can all be defined solely by means of the metric, in direct analogy to similar concepts within $\mathbb{R}^n$.

Main sources for this article were [1, 2, 3] and [4]. Also heavily used was [5], which provides a somewhat alternative approach to the subject with emphasis on applications in general relativity theory.

1.1 Metric tensor

A **metric tensor** (or simply **metric**) on a $\mathcal{C}^\infty$-manifold M is a 2-covariant, symmetric, non-degenerate tensor field g (often noted ds^2), introducing the notion of a scalar product as an intrinsic property of the manifold[2]. Non-degeneracy refers to the condition that, for any vector $0 \neq \mathrm{v} \in T_pM$, the 1-form $g(\mathrm{v}, \cdot)$ is not trivial on T_pM. A manifold M equipped with a metric tensor is called a **generalized Riemannian manifold**, noted (M, g). Note how the usual positive-definitness of scalar products is replaced by the weaker condition of non-degeneracy. In case of a positive-definit metric, (M, g) is called a **Riemannian manifold**. Being a tensor of rang 2, the metric can be written in local coordinates in the form

$$g = g_{ij} \cdot dx^i dx^j, \tag{1.1}$$

with $g_{ij} \in \mathcal{C}^\infty(M, \mathbb{R})$ generally as non-constant, real functions given by $g_{ij} = g(\partial_i, \partial_j)$. With the metric tensor $(g_{ij}) = \mathrm{diag}(1, -1, -1, -1)$, that is,

$$g = dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2, \tag{1.2}$$

one arrives at the special case of the **Minkowski space**, known from special theory of relativity. The **norm** of a vector $\mathrm{v} = \mathrm{v}^i \partial_i \in T_pM$ at some point $p \in M$ is defined to be

$$\|\mathrm{v}\| := \sqrt{|g(\mathrm{v}, \mathrm{v})|} = \sqrt{|g_{ij} \cdot \mathrm{v}^i \mathrm{v}^j|}. \tag{1.3}$$

Similarly, two vectors $u, \mathrm{v} \in T_pM$ are called **orthogonal** if they satisfy $g(u, \mathrm{v}) = 0$. Both the norm and orthogonality can readily be seen as direct generalizations of the same concepts within $(\mathbb{R}^n, \langle, \rangle)$.

1.2 The pullback metric and embedded manifolds

An important class of manifolds are manifolds embedded in $\mathbb{R}^n$, *inheriting* their metric from their super-space. A formalization of this concept is given via pushing forward vectors and pulling back tensors.

Let M, N be two $\mathcal{C}^\infty$-manifolds and $f : M \to N$ a smooth mapping. Then for each fixed point $p \in T_pM$, the function f induces a natural linear mapping $df : T_pM \to T_{f(p)}N$ between the two tangent-spaces T_pM and $T_{f(p)}N$ via $\mathrm{v} \mapsto df(\mathrm{v})$, whereas $df(\mathrm{v}) \in T_{f(p)}N$ is the vector satisfying $df(\mathrm{v})h = \mathrm{v}(h \circ f)$ for any smooth function $h \in \mathcal{C}^\infty(M, \mathbb{R})$. The vector $df(\mathrm{v})$, called the **pushforward**[6] of v by f, can be interpreted as directional derivative of f along the vector v at the point $p \in M$. The mapping df its self thus corresponds to the total derivative of f. If $\{x_M^i\}$ and $\{x_N^i\}$ are local coordinates around $p \in M$ and $f(p) \in N$ respectively, then df satisfies

$$df(\partial_{x_M^i}) = \frac{\partial f^j}{\partial x_M^i} \cdot \partial_{x_N^j}. \tag{1.4}$$

In the case that N is equipped with a metric tensor g, the mapping df can be used to define a metric f^*g on M by *pushing forward* any vectors from the tangent bundle TM to the tangent bundle TN. The metric f^*g defined by

$$(f^*g)(u, \mathrm{v})\big|_p := g(df(u), df(\mathrm{v}))\big|_{f(p)} \quad , \quad u, \mathrm{v} \in T_pM, \ p \in M \tag{1.5}$$

is called the **pullback metric**[6] of N to M via f. Its components are given in local coordinates by

$$(f^*g)_{ij} = (f^*g)\big(\partial_{x^i_M}, \partial_{x^j_M}\big) \overset{(1.4)}{=} \frac{\partial f^k}{\partial x^i_M} \frac{\partial f^l}{\partial x^j_M} \cdot g_{kl}. \tag{1.6}$$

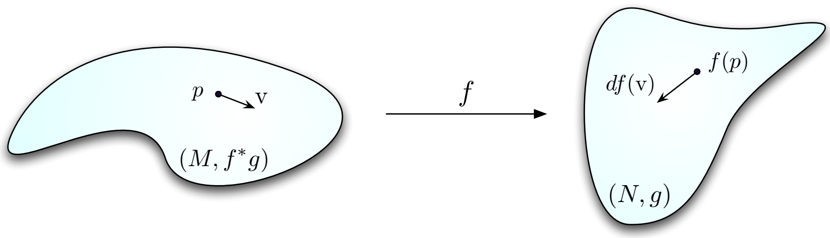

Figure 1.1: On the definition of pushforward vectors and pullback metrics.

As an example, consider a $\mathcal{C}^\infty$-manifold M embedded into $\mathbb{R}^n$ via the embedding $J : M \hookrightarrow \mathbb{R}^n$. Then the euclidean scalar product $g := \langle \cdot, \cdot \rangle$, a special case of a metric tensor on the Riemannian manifold $\mathbb{R}^n$, induces on M the pullback metric $J^*g\,(u, \mathrm{v}) := \langle dJ(u), dJ(\mathrm{v}) \rangle$ with components

$$(J^*g)_{ij} = \sum_k \frac{\partial J^k}{\partial x^i_M} \frac{\partial J^k}{\partial x^j_M} \tag{1.7}$$

in local coordinates.

1.3 The signature of the metric tensor

Restricted to the tangent space of a fixed point $p \in M$, the metric is nothing but a symmetric, non-degenerate, bilinear form defined on an n-dimensional space. By Sylvester's law of inertia[7] there exists a basis in T_pM so that $g\big|_p$ takes a diagonal form, with only $+1$s and -1s on the diagonal. A basis like that is called **orthonormal**, while the number n_+ and n_- of $+1$s and -1s respectively is independent of the chosen orthonormal basis. The tuple (n_+, n_-) is referred to as the signature of the metric. For space-time manifolds in general relativity, typical signatures are $(3, 1)$ or $(1, 3)$, with the single -1 (or $+1$) corresponding to the so called **time-like subspace**[8].

Furthermore, by means of linear coordinate transformations, one can always find a local chart on the manifold whose coordinate vectors form on a given point $p \in M$ an orthonormal basis. As we will see later on in 3.3, a suitable choice of coordinates even leads to vanishing first derivatives of the metric components at point p.

2 Covariant derivative

Applications of differential geometry to physical problems often require differentiation of scalar, vector and even higher-rank tensor fields, which translates to a comparison of *nearby* tangent spaces and their duals. As there exists à priori no natural way of comparing tangent spaces at different points, vectors need to be *transported* from one tangent space to the other, typically along some curve and in some coordinate-invariant manner. Such means of transport are provided by so called *connections*, presented below.

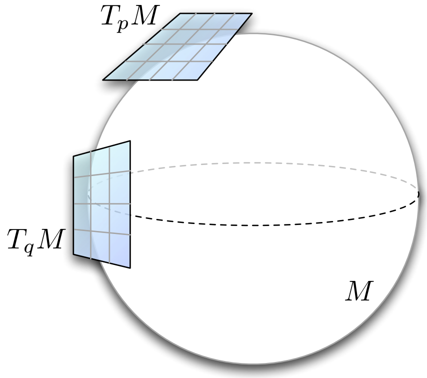

Figure 2.1: On the comparison of tangent spaces on different points: À priori impossible.

2.1 The covariant derivative & parallel transport

Let M be a $\mathcal{C}^\infty$-manifold, $\mathcal{C}^\infty(M,\mathbb{R})$ the linear space of real, smooth functions on M and $\mathcal{C}^\infty(M,TM)$ the linear space of all smooth vector fields on M. A **connection**[5] on M is a mapping

$$\mathcal{C}^\infty(M,TM)\times\mathcal{C}^\infty(M,TM)\to\mathcal{C}^\infty(M,TM)\quad,\quad (X,Y)\mapsto\nabla_X Y \tag{2.1}$$

satisfying for all $X,Y,Z\in\mathcal{C}^\infty(M,TM)$, $\lambda\in\mathbb{R}$ and $f\in\mathcal{C}^\infty(M,\mathbb{R})$ the following properties:

D01. ∇ is $\mathbb{R}$- and $\mathcal{C}^\infty(M,\mathbb{R})$-linear in its first variable, that is, $\nabla_{fX+\lambda Y}Z=f\nabla_X Z+\lambda\nabla_Y Z$.

D02. ∇ is $\mathbb{R}$-linear in its second variable, that is, $\nabla_X(\lambda Y+Z)=\lambda\nabla_X Y+\nabla_X Z$.

D03. ∇ satisfies the Leibniz rule in the second variable, that is, $\nabla_X(fY)=f\nabla_X Y+(Xf)\cdot Y$.

It can be shown that $\nabla_X Y\big|_p$ depends only on the vector $X\big|_p$ and the field Y in an arbitrarily small neighborhood of p. Thus, $\nabla_X Y\big|_p$ can be interpreted as a kind of derivative of the field Y along the vector $X\big|_p$, yielding a new, intrinsic vector at p, called the **covariant derivative** of Y along $X\big|_p$. As a generalization of the known product rule for derivatives of q-linear forms of vector fields in $\mathbb{R}^n$, one defines the covariant derivative of a q-covariant tensor field T by demanding

$$\nabla_X\left(T(Y_1,..,Y_q)\right)\overset{!}{=}(\nabla_X T)(Y_1,..,Y_q)+\sum_{i=1}^{q}T\left(Y_1,..,Y_{i-1},\nabla_X Y_i,Y_{i+1},..,Y_q\right) \tag{2.2}$$

for all vector fields $X,Y_1,..,Y_q\in\mathcal{C}^\infty(M,TM)$, whereas by convention $\nabla_X f:=Xf$ for any smooth scalar field $f\in\mathcal{C}^\infty(M,\mathbb{R})$. In the same exact way one defines the covariant derivative of contravariant tensors. Due to linearity of the covariant derivative, the covariant derivatives of coordinate vector fields along coordinate vectors are of central importance. The corresponding coefficients Γ^r_{ij}, smooth functions on M defined by

$$\nabla_{\partial_i}\partial_j=:\Gamma^r_{ij}\partial_r, \tag{2.3}$$

are called the **Christoffel symbols** of the connection. Using definition (2.2) and (2.3), it is straightforward to show that in local coordinates the covariant derivative of a p-contravariant, q-covariant tensor T can be written by means of the Christoffel-symbols as

$$(\nabla_{\partial_r}T)^{i_1..i_p}_{j_1..j_q}=\partial_r T^{i_1..i_p}_{j_1..j_q}+\Gamma^{i_1}_{si}T^{si_2..i_p}_{j_1..j_q}+\cdots+\Gamma^{i_p}_{si}T^{i_1..i_{p-1}s}_{j_1..j_q}-\Gamma^s_{j_1i}T^{i_1..i_p}_{sj_2..j_q}-\cdots-\Gamma^s_{j_qi}T^{i_1..i_p}_{j_1..j_{q-1}s}. \tag{2.4}$$

The presence of a connection permits to introduce the notion of **parallel transport** of a vector along a curve: One says that the vector field X defined on the curve $\gamma:I\subseteq\mathbb{R}\to M$ is the **parallel transport** of the vector $\mathrm{v}\in T_pM$ (with $p\in\gamma(I)$) along γ if $X_p=\mathrm{v}$ and $\nabla_{\dot\gamma(t)}X=0$ for all $t\in I$. These conditions correspond to the ordinary differential equation

$$\dot\gamma^i\partial_i X^j+\dot\gamma^i\Gamma^j_{ir}X^r=0 \tag{2.5}$$

with initial value $X^r\big|_p=\mathrm{v}^r$, uniquely defining X along the curve. By fixing a curve $\gamma:[0,1]\to M$ between two points $p,q\in M$, one obtains by means of parallel transport an isomorphism from the tangent space T_pM to the tangent space T_qM, a so called *connection* (hence the name)[5, 2].

2.2 Definition: Riemannian connection

A connection ∇ on a generalized Riemannian manifold (M, g) is called a **Riemannian connection**[2] if it satisfies the following additional constraints:

L01. **Metric compatibility:** The induced parallel translations are isometries between tangent spaces, that is, $g(u, v) = g(\widetilde{u}, \widetilde{v})$ for any vectors $u, v \in T_pM$, parallel-transported along some curve γ to $\widetilde{u}, \widetilde{v} \in T_qM$ respectively.

L02. **Vanishing torsion:** The connection is **torsion free**, that is $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i = 0$ for all i, j. Notice the similarity to Schwarz's theorem about partial derivatives in $\mathbb{R}^n$.

Note that by applying rule (2.2) to the condition $\nabla_{\dot{\gamma}}(g(U, V)) = 0$, with U & V being the parallel transports of u and v respectively along γ, one finds (L01) to be equivalent to $\nabla_X g = 0\ \forall X \in TM$. Also note that (L02) is equivalent to the 1-contravariant, 2-covariant **torsion tensor**

$$T_{\text{tor}}(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.6}$$

to be vanishing for arbitrary vector fields X, Y.

By the fundamental theorem of Riemannian geometry, for any generalized Riemannian manifold (M, g) there exists one unique Riemannian connection ∇. We shall briefly present here a proof found in literature[5]. Let X, Y, Z be any arbitrary vector fields on M and let ∇ be some Riemannian connection on M. By condition $\nabla_X g = 0$ one has

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{2.7}$$

by condition (L02) one has

$$\nabla_X Z = \nabla_Z X + [X, Z]\,. \tag{2.8}$$

Using (2.8) in (2.7) one obtains

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_Z X) + g(Y, [X, Z]). \tag{2.9}$$

In the same way one obtains

$$Yg(Z, X) = g(\nabla_Y Z, X) + g(Z, \nabla_X Y) + g(Z, [Y, X]) \tag{2.10}$$

and

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Y Z) + g(X, [Z, Y]). \tag{2.11}$$

Adding (2.10) to and subtracting (2.11) from (2.9) one finds the so called **Koszul-formula**

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \tag{2.12}$$

The left hand side of (2.12) is, for fixed X, Y, a 1-covariant tensor field in the variable Z, determined by the right hand side of (2.12). As the metric is non-degenerate, this uniquely determines the vector field $\nabla_X Y$ and proves the existence and unicity of the Riemannian connection. It can easily be shown that this so defined operation ∇ is a Riemannian connection[5]. Finally, setting $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$ and using (2.3) in the Koszul-formula (2.12) yields

$$2\Gamma^l_{ij} g_{lk} = 2g\left(\nabla_{\partial_i}\partial_j, \partial_k\right) = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \tag{2.13}$$

and thus the Christoffel-symbols

$$\boxed{\Gamma^r_{ij} = \frac{g^{rk}}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)} \tag{2.14}$$

for the unique Riemannian connection.

2.3 Geometrical interpretation of the Riemannian connection

Consider the special case of an m-dimensional manifold M embedded into $\mathbb{R}^n$ via the embedding $J : M \hookrightarrow \mathbb{R}^n$. Suppose on M the metric g as the pullback metric of the euclidean metric in $\mathbb{R}^n$. Let us fix a point $p \in M$ and suppose w.l.o.g. the local coordinates on that point so that $g(\partial_i, \partial_j)\big|_p = \delta_{ij}$. Let us for $k \in \{1, .., n\}$ and $i \in \{1, .., m\}$ note $\partial_i^k := [dJ(\partial_i)]^k$ the k-component of the pushforward of ∂_i. Then

$$\partial_s g_{ij} = \partial_s \langle dJ(\partial_i), dJ(\partial_j)\rangle = \partial_s \sum_{k=1}^n \partial_i^k \partial_j^k = \sum_{k=1}^n \left[(\partial_s \partial_i^k) \cdot \partial_j^k + \partial_i^k \cdot (\partial_s \partial_j^k)\right]. \tag{2.15}$$

The Christoffel-symbols of the Riemannian connection are by (2.14) and (2.15) given at p by

$$\begin{aligned}
\Gamma_{ij}^r\big|_p &= \frac{\delta^{rs}}{2}\left(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij}\right)\\
&= \frac{1}{2}\sum_{k=1}^n \Big[(\partial_i \partial_j^k)\cdot \partial_r^k + \cancel{\partial_j^k \cdot (\partial_i \partial_r^k)} + (\partial_j \partial_i^k)\cdot \partial_r^k + \cancel{\partial_i^k \cdot (\partial_j \partial_r^k)} - \cancel{(\partial_r \partial_i^k)\cdot \partial_j^k} - \cancel{\partial_i^k \cdot (\partial_r \partial_j^k)}\Big]\\
&= \sum_{k=1}^n (\partial_i \partial_j^k)\cdot \partial_r^k,
\end{aligned} \tag{2.16}$$

whereas in the last step we used the fact that

$$\partial_i \partial_j^k = \partial_i \left[dJ(\partial_j)\right]^k \overset{(1.4)}{=} \partial_i \partial_j J^k = \partial_j \partial_i J^k \overset{(1.4)}{=} \partial_j \partial_i^k. \tag{2.17}$$

Now let $X \in T_pM$ be some vector and $Y \in \mathcal{C}^\infty(M, TM)$ some vector field. By (2.4) and (2.16) one obtains

$$\begin{aligned}
\nabla_X Y\big|_p &= X^i \nabla_{\partial_i} Y = X^i(\partial_i Y^r)\partial_r + X^i Y^j \Gamma_{ij}^r \partial_r\\
&= X^i(\partial_i Y^r)\partial_r + X^i Y^j \sum_{k=1}^n \sum_{r=1}^m (\partial_i \partial_j^k)\cdot \partial_r^k \cdot \partial_r
\end{aligned} \tag{2.18}$$

as covariant derivative of Y along X at p, per se a tangent vector on the manifold. On the other hand, let us regard $dJ(Y)$ as a field of vectors in $\mathbb{R}^n$ defined on M. Its directional derivative along X is then given by

$$\left[D_X dJ(Y)\right]^k = X^i \partial_i \left[dJ(Y)\right]^k = X^i \partial_i \left[Y^j \partial_j^k\right] = X^i(\partial_i Y^j)\partial_j^k + X^i Y^j \cdot (\partial_i \partial_j^k), \tag{2.19}$$

for $k \in \{1, .., n\}$. Let $P_p : T_p\mathbb{R}^n \to T_pM$ be the orthogonal projection on the manifold's tangent space, given by

$$P_p(Z) := \sum_{r=1}^m \langle Z, dJ(\partial_r)\rangle \partial_r = \sum_{r=1}^m \sum_{k=1}^n Z^k \partial_r^k \cdot \partial_r. \tag{2.20}$$

Then

$$P_p\left[D_X dJ(Y)\big|_p\right] = \sum_{r=1}^m \sum_{k=1}^n \left[X^i(\partial_i Y^j)\partial_j^k + X^i Y^j \cdot (\partial_i \partial_j^k)\right]\cdot \partial_r^k \cdot \partial_r \tag{2.21}$$

$$\begin{aligned}
&= \sum_{r=1}^m X^i(\partial_i Y^j) \underbrace{\sum_{k=1}^n \partial_j^k \partial_r^k}_{\substack{g(\partial_j, \partial_r)\\ =\delta_{jr}}} \cdot \partial_r + \sum_{r=1}^m \sum_{k=1}^n X^i Y^j \cdot (\partial_i \partial_j^k)\cdot \partial_r^k \cdot \partial_r\\
&= X^i(\partial_i Y^j)\cdot \partial_j + X^i Y^j \sum_{k=1}^n \sum_{r=1}^m (\partial_i \partial_j^k)\cdot \partial_r^k \cdot \partial_r\\
&\overset{(2.18)}{=} \nabla_X Y\big|_p,
\end{aligned} \tag{2.22}$$

that is, the Riemannian covariant derivative $\nabla_X Y\big|_p$ can be viewed as the orthogonal projection of the derivative of the vector field Y in $\mathbb{R}^n$ along X, to the tangent space T_pM of the manifold.

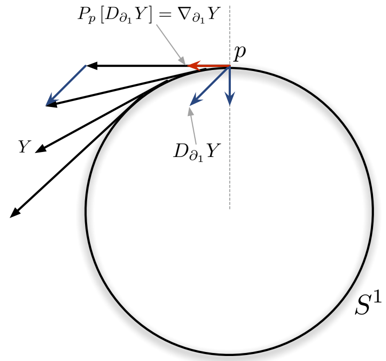

Figure 2.2: On the Riemannian covariant derivative $\nabla_{\partial_1} Y$ on S^1 as a projection of the directional derivative in $\mathbb{R}^n$ to the tangent space T_pS^1.

If $\gamma : \mathbb{R} \to M$ is a smooth curve on the manifold, then $\nabla_{\dot\gamma} Y = 0$ means that the orthogonal projection of the rate of change of Y along γ onto the tangent space of the manifold vanishes. In particular, $\nabla_{\dot\gamma}\dot\gamma = 0$ means that the projection of the rate of change of the velocity $\dot\gamma$ along γ, to the tangent space of the manifold, vanishes. Curves of this kind are called **geodesics**, as we shall see shortly below.

3 Geodesics

Geodesics generalize the concept of *straight lines* in $\mathbb{R}^n$ to abstract manifolds by means of parallel transport or by means of distance minimization. As we shall see, both approaches are equivalent in the case of Riemannian connections. Geodesics are of central importance in general theory of relativity, as they formalize the concept of a free falling object within a gravitational field. Furthermore, inertial frames known from special relativity take in general relativity the form of local inertial frames, falling freely under the influence solely of the *background* gravitational field and in which physics can be locally described as in gravitationless flat space. The coordinates used to describe such inertial frames fall under the term *normal coordinates*, defined using the so called *exponential map* as illustrated in 3.3.

3.1 Geodesics of the connection

Let M be a $\mathcal{C}^\infty$-manifold, equipped with a connection ∇. Then a curve $x : I \subseteq \mathbb{R} \to M$ is called **geodesic of the connection**[9] ∇ if it satisfies

$$\nabla_{\dot x}\dot x = 0, \tag{3.1}$$

that is, its tangent vector is parallel-transported along the curve. Using the axioms in 2.1 for ∇, it follows readily that (3.1) is equivalent to the ordinary differential equation

$$\boxed{\ddot x^r + \dot x^i \dot x^j \Gamma^r_{ij} = 0.} \tag{3.2}$$

As already seen in 2.3, condition (3.1) can be interpreted as the velocity of the curve not changing with *time*, thus describing in a sense the trajectory of a non-accelerating point on the manifold.

3.2 Geodesics of the metric

Note how the notion of geodesics presented in 3.1 is by its nature independent of the existence or not of an underlying metric, only supposing the existence of a connection. For generalized Riemannian manifolds, there exists an alternative approach to defining geodesics by means of curve length, leading to the same notion resulting from the existing Riemannian connection. The length of a smooth, time-like[1] curve $x : [a, b] \to M$ on

[1]Here defined to satisfy $g(\dot x, \dot x) > 0$.

a pseudo-Riemannian manifold (M, g) is defined to be

$$L(\gamma) := \int_a^b \sqrt{g(\dot{x}(t), \dot{x}(t))}\, dt, \tag{3.3}$$

which can easily be verified to be independent of its parametrization[5]. One calls a **(time-like) geodesic of the metric** g a (time-like) curve locally extremizing the curve-length between any two subsequent, adequately nearby points. Considering $l : (x, y) \mapsto \sqrt{g(y, y)\big|_x}$ as a Lagrange density on the manifold's tangent bundle TM, one arrives by applying this variational principle to (3.3) at the known Euler-Lagrange equations[4]

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}^k} l(x, \dot{x}) - \frac{\partial}{\partial x^k} l(x, \dot{x}) = 0 \tag{3.4}$$

in local coordinates for any geodesic $x : [a, b] \to M$. For time-like curves one has at all times $g(\dot{x}, \dot{x}) > 0$, so we can w.l.o.g. suppose via re-parametrization that $g(\dot{x}, \dot{x})$ is constant along the geodesic. One then finds for (3.4) the special form

$$-\frac{1}{2}(\partial_k g_{ij})\dot{x}^i \dot{x}^j + g_{kj}\ddot{x}^j + \dot{x}^j \dot{x}^l \partial_l g_{kj} = 0, \tag{3.5}$$

which by contraction with g^{rk} leads to the system of differential equations

$$\ddot{x}^r + \frac{g^{rk}}{2}\left[\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}\right] \cdot \dot{x}^i \dot{x}^j = 0. \tag{3.6}$$

Using (2.14) one finds (3.6) to be totally equivalent to (3.2), that is,

$$\ddot{x}^r + \dot{x}^i \dot{x}^j \Gamma^r_{ij} = 0. \tag{3.7}$$

Any time-like geodesic of the metric with constant non-zero speed is thus a geodesic of the Riemannian connection. Inversely, as the later is by axiom (L01) an isometry, any geodesic of the Riemannian connection satisfies

$$\nabla_{\dot{x}}(g(\dot{x}, \dot{x})) = \underbrace{(\nabla_{\dot{x}} g)}_{0}(\dot{x}, \dot{x}) + g(\underbrace{\nabla_{\dot{x}}\dot{x}}_{0}, \dot{x}) + g(\dot{x}, \underbrace{\nabla_{\dot{x}}\dot{x}}_{0}) = 0, \tag{3.8}$$

that is, has constant speed.

It is noteworthy that one directly arrives at (3.5) and thus (3.7) by formally applying the Euler-Lagrange equations (3.4) to the Lagrange function $\mathcal{L}(x, \dot{x}) := g(\dot{x}, \dot{x})\big|_x$, independently of any assumptions on the speed. For manifolds embedded into $\mathbb{R}^3$ with the euclidean metric, this corresponds up to a constant factor to the kinetic energy of a moving particle constrained to the manifold and to its Lagrangian in case of a free particle. Geodesics can thus be seen as trajectories of free particles merely confined to the manifold.

In general relativity theory, free falling objects simply follow the geodesics in space time. Gravitation is no longer seen as a *force field*, but simply an aspect of space-time structure, determining the rules for which curve gets to be a geodesic and which not[4].

3.3 The exponential map, normal coordinates and local reference frames

Equation (3.7) is a non-linear, ordinary differential equation of second degree, describing the development of a curve by means of its tangent vectors and their change. For any given start-vector $\dot{x}(0) =: \mathrm{v}$ at point $x(0) =: x_o$, there exists to it a unique, local solution[10], that is, a geodesic $x : [0, \varepsilon] \to M$ starting at x_o with initial tangent vector v. Note that for any $\lambda \geq 0$, the curve $y(t) := x(\lambda t)$ is also a solution of (3.7), and is the unique geodesic starting at point x_o with initial velocity $\lambda\mathrm{v}$.

Mapping the vector v to the point $x(1)$ reached by the generated geodesic $x : [0, \infty) \to M$ after unit-time, one obtains the so-called **exponential map** $\exp_{x_o} : U \subseteq T_{x_o}M \to M$, mapping a neighborhood U of the origin of the tangential space $T_{x_o}M$ at $x_o \in M$ to the manifold M[4].

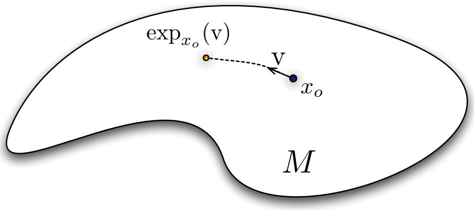

Figure 3.1: On the definition of the exponential map, mapping the tangential space $T_{x_o}M$ to M by means of geodesics.

In local coordinates, the exponential map is given as a solution of (3.7) for unit time by

$$\left[\exp_{x_o}(\mathrm{v})\right]^r = x_o^r + \mathrm{v}^r - \frac{1}{2}\Gamma^r_{ij}\big|_{x_o}\mathrm{v}^i\mathrm{v}^j + \mathcal{O}(\mathrm{v}^3). \tag{3.9}$$

As the Jacobi-determinant of (3.9) shows, $\exp_{x_o} : \mathrm{v} \mapsto \exp_{x_0}(\mathrm{v}) = x(\mathrm{v})$ is locally a diffeomorphism at the origin of $T_{x_o}M$. Fixing some base in $T_{x_o}M$ (e.g. $\{\partial_i\}_i$) we can identify $T_{x_o}M$ with $\mathbb{R}^n$, thus effectively obtaining with $\exp^{-1}_{x_o}$ a local chart of a neighborhood of x_o into $\mathbb{R}^n$. The corresponding coordinates are called **normal coordinates**[4, 11]. In these coordinates the metric takes the form

$$\begin{aligned} g(\partial_{\mathrm{v}^i}, \partial_{\mathrm{v}^j}) = \frac{\partial x^k}{\partial \mathrm{v}^i}\frac{\partial x^l}{\partial \mathrm{v}^j}\cdot g_{kl} &= \left(\delta^k_i - \Gamma^k_{ir}\big|_{x_o}\mathrm{v}^r + \mathcal{O}(\mathrm{v}^2)\right)\cdot\left(\delta^l_j - \Gamma^l_{jr}\big|_{x_o}\mathrm{v}^r + \mathcal{O}(\mathrm{v}^2)\right)\cdot g_{kl} \\ &= g_{ij} - \Gamma^k_{ir}\big|_{x_o}g_{kj}\mathrm{v}^r - \Gamma^l_{jr}\big|_{x_o}g_{il}\mathrm{v}^r + \mathcal{O}(\mathrm{v}^2) = g_{ij} - \mathrm{v}^r\cdot\partial_r g_{ij}\big|_{x_o} + \mathcal{O}(\mathrm{v}^2) \\ &= g_{ij}\big|_{x_o} + \mathcal{O}(\mathrm{v}^2). \end{aligned} \tag{3.10}$$

As we can w.l.o.g. suppose the base $\{\partial_{x^i}\}_i$ to be orthonormal[2] in x_o, in normal coordinates the metric becomes diagonal with only 1s and -1s on the diagonal up to 1st order around x_o. For the special case of a Lorentzian manifold it takes the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \mathcal{O}(\mathrm{v}^2). \tag{3.11}$$

It corresponds to a locally flat spacetime, in which all Christoffel symbols vanish at x_o and the geodesic equation (3.7) becomes the equation of a forceless non-accelerating body in a locally flat spacetime. Covariant derivatives take the form of simple partial derivatives, so that physics locally reduces in these coordinates to flat-space inertial frame physics. By Einsteins famous *equivalence principle*, such a reference frame is the reference frame of an observer freely accelerated within the *background* gravitational field[4, 12].

4 Example: The surface of the unit sphere

We shall illustrate some of the above presented notions for the surface $S^2 \overset{J}{\hookrightarrow} \mathbb{R}^3$ of the unit-sphere naturally embedded into $\mathbb{R}^3$ via the inclusion J, equipped with the pullback euclidean metric g. Using the local chart

$$(0,\pi)\times(0,2\pi) \to S^2 \quad , \quad (\vartheta,\varphi) \mapsto \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}, \tag{4.1}$$

one finds the corresponding (pushforward) coordinate vectors via (1.4) to be

$$dJ(\partial_\vartheta) = \begin{pmatrix} \cos\vartheta\cos\varphi \\ \cos\vartheta\sin\varphi \\ -\sin\vartheta \end{pmatrix} \quad , \quad dJ(\partial_\varphi) = \begin{pmatrix} -\sin\vartheta\sin\varphi \\ \sin\vartheta\cos\varphi \\ 0 \end{pmatrix} \quad . \tag{4.2}$$

[2]For example by a simple linear transformation of coordinates.

Using the definition of the pullback metric, one finds the components of the metric to take the form

$$(g_{ij}) = \begin{pmatrix} \langle dJ(\partial_\vartheta), dJ(\partial_\vartheta)\rangle & \langle dJ(\partial_\vartheta), dJ(\partial_\varphi)\rangle \\ \langle dJ(\partial_\varphi), dJ(\partial_\vartheta)\rangle & \langle dJ(\partial_\varphi), dJ(\partial_\varphi)\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{pmatrix}. \tag{4.3}$$

Applying (2.14) to (4.3), one finds the Christoffel symbols of the Riemannian connection to be

$$\begin{aligned} \Gamma^\vartheta_{\varphi\varphi} &= -\sin\vartheta\cos\vartheta, \\ \Gamma^\varphi_{\varphi\vartheta} = \Gamma^\varphi_{\vartheta\varphi} &= \frac{\cos\vartheta}{\sin\vartheta}, \\ \Gamma^k_{ij} &= 0 \ \text{ : otherwise.} \end{aligned} \tag{4.4}$$

The geodesic equations (3.7) subsequently take the form

$$\ddot\varphi = -\dot\varphi\dot\vartheta\cdot\frac{\cos\vartheta}{\sin\vartheta}\ , \quad \ddot\vartheta = \dot\varphi^2\cdot\sin\vartheta\cos\vartheta. \tag{4.5}$$

For the start-point $x_o = (\vartheta_o, \varphi_o) = (\pi/2, 0)$ and start-vector $\mathrm{v} = (0, \omega)$, the resulting geodesic is given by $x(t) = (\vartheta(t), \varphi(t)) = (\pi/2, \omega t)$, corresponding to the *equator* of the sphere. In particular, $\exp_{(\pi/2,0)}(0,\omega) = (\pi/2, \omega)$.

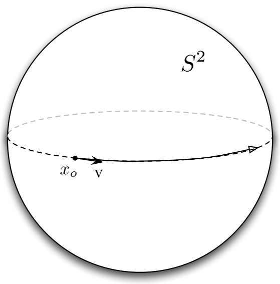

Figure 4.1: Example geodesic on S^2: The geodesics on S^2 are nothing else than the great circles, locally minimizing the traversed distance between nearby consecutive points.

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