Quantenmechanik II
FSU Jena - WS 2009/2010
Übungsserie 06 - Lösungen

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Aufgabe 12

Mit

$$\mathbf{B} := \nabla \times \mathbf{A} \quad , \quad \mathbf{E} := -\nabla\Phi - \dot{\mathbf{A}}$$

als tatsächliche physikalische Felder, ist ersichtlich dass für beliebiges Skalarfeld $U(t,\mathbf{x}),\ U \in \mathcal{C}^2$ auch

$$\mathbf{A}' := \mathbf{A} + \nabla U \quad , \quad \Phi' := \Phi - \dot{U}$$

die gleichen $\mathbf{B},\ \mathbf{E}$ -Felder definieren, denn

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \underbrace{\nabla \times (\nabla U)}_{0} = \mathbf{B}$$

$$-\nabla\Phi' - \dot{\mathbf{A}}' = -\nabla\Phi - \dot{\mathbf{A}} + \cancel{\nabla\partial_t U} - \cancel{\partial_t \nabla U}$$

Gesucht ist nun ein geeignetes U mit $\nabla \cdot \mathbf{A}' = 0 = \Phi'$. Wegen

$$\partial_t \left[\nabla \cdot \mathbf{A}(t,\mathbf{x}) + \int_0^t \Delta\Phi(\tau,\mathbf{x})\ d\tau \right] = -\nabla \cdot \underbrace{\partial_t \left[-\mathbf{A}(t,\mathbf{x}) - \int_0^t \nabla\Phi(\tau,\mathbf{x})\ d\tau \right]}_{\mathbf{E}(t,\mathbf{x})} = -\nabla \cdot \mathbf{E}(t,\mathbf{x}) \overset{\text{Vakuum}}{=} 0$$

ist $\nabla \cdot \mathbf{A}(t,\mathbf{x}) + \int_0^t \Delta\Phi(\tau,\mathbf{x})\ d\tau$ Zeitunabhängig. Dabei ist die Poissongleichung

$$\Delta h(\mathbf{x}) = -\left[\nabla \cdot \mathbf{A}(t,\mathbf{x}) + \int_0^t \Delta\Phi(\tau,\mathbf{x})\ d\tau \right] \tag{0.1}$$

(bei hinreichend glatten Feldern) stets lösbar und der Ansatz

$$U(t,\mathbf{x}) := \int_0^t \Phi(\tau,\mathbf{x})\ d\tau + h(\mathbf{x}) \tag{0.2}$$

erfüllt genau die obigen Forderungen, denn

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \Delta U \overset{(0.2)}{=} \nabla \cdot \mathbf{A} + \int \Delta\Phi\ dt + \Delta h \overset{(0.1)}{=} 0$$

$$\Phi' = \Phi - \partial_t U \overset{(0.2)}{=} 0$$

□

Aufgabe 13

Beginnen mit der Definition

$$|n_1, .., n_{\boldsymbol{\mu}}, ..\rangle := \ldots \frac{(\hat{a}^\dagger_{\boldsymbol{\mu}})^{n_{\boldsymbol{\mu}}}}{\sqrt{n_{\boldsymbol{\mu}}!}} \ldots \frac{(\hat{a}^\dagger_1)^{n_1}}{\sqrt{n_1!}} |0\rangle$$

und

$$\hat{\mathbf{E}}(t, \mathbf{x}) := -\partial_t \hat{\mathbf{A}}(t, \mathbf{x}) = \sum_{\boldsymbol{\mu}} \left(\frac{\hbar \omega_{\boldsymbol{\mu}}}{2V\varepsilon_0} \right)^{\frac{1}{2}} \mathbf{e}_{\boldsymbol{\mu}} \cdot i \cdot \left[e^{i\mathbf{k}_{\boldsymbol{\mu}}\mathbf{x} - i\omega_{\boldsymbol{\mu}} t} \hat{a}_{\boldsymbol{\mu}} - e^{-i\mathbf{k}_{\boldsymbol{\mu}}\mathbf{x} + i\omega_{\boldsymbol{\mu}} t} \hat{a}^\dagger_{\boldsymbol{\mu}} \right]$$

$$\hat{\mathbf{B}}(t, \mathbf{x}) := \nabla \times \hat{\mathbf{A}}(t, \mathbf{x}) = \sum_{\boldsymbol{\mu}} \left(\frac{\hbar}{2V\varepsilon_0 \omega_{\boldsymbol{\mu}}} \right)^{\frac{1}{2}} (\mathbf{k}_{\boldsymbol{\mu}} \times \mathbf{e}_{\boldsymbol{\mu}}) \cdot i \cdot \left[e^{i\mathbf{k}_{\boldsymbol{\mu}}\mathbf{x} - i\omega_{\boldsymbol{\mu}} t} \hat{a}_{\boldsymbol{\mu}} - e^{-i\mathbf{k}_{\boldsymbol{\mu}}\mathbf{x} + i\omega_{\boldsymbol{\mu}} t} \hat{a}^\dagger_{\boldsymbol{\mu}} \right]$$

wobei

$$\boldsymbol{\mu} = (\underbrace{\mu_1, \mu_2, \mu_3}_{\mathbf{m}}, \mu_4) \in \mathbb{Z}^3 \times \{\pm 1\} \quad , \quad \mathbf{k}_{\boldsymbol{\mu}} = \mathbf{k}_{\mathbf{m}} = \frac{2\pi}{L} \mathbf{m}$$

$$\mathbf{e}_{\boldsymbol{\mu}} \perp \mathbf{k}_{\boldsymbol{\mu}} \quad , \quad \mathbf{e}_{(\mathbf{m},1)} \perp \mathbf{e}_{(\mathbf{m},2)} \quad , \quad \omega_{\boldsymbol{\mu}} = \omega_{\mathbf{m}} = c \, \|\mathbf{k}_{\mathbf{m}}\| \quad , \quad \mathbf{e}_{(-\mathbf{m},p)} \overset{\text{o.B.d.A.}}{=} \mathbf{e}_{(\mathbf{m},p)}, \; p = \pm 1$$

und schreiben

$$\hat{\mathbf{p}}_S = \frac{1}{c^2\mu_0}\int_V d\mathbf{x}\; \hat{\mathbf{E}} \times \hat{\mathbf{B}}$$

$$= -\frac{\hbar}{2V}\int_V d\mathbf{x} \sum_{\boldsymbol{\mu},\boldsymbol{\nu}\in\mathbb{Z}^3\times\{\pm 1\}} \mathbf{e}_{\boldsymbol{\mu}} \times (\mathbf{k}_{\boldsymbol{\nu}} \times \mathbf{e}_{\boldsymbol{\nu}}) \cdot \left[e^{i\mathbf{k}_{\boldsymbol{\mu}}\mathbf{x}-i\omega_{\boldsymbol{\mu}}t}\hat{a}_{\boldsymbol{\mu}} - e^{-i\mathbf{k}_{\boldsymbol{\mu}}\mathbf{x}+i\omega_{\boldsymbol{\mu}}t}\hat{a}^\dagger_{\boldsymbol{\mu}}\right] \circ \left[e^{i\mathbf{k}_{\boldsymbol{\nu}}\mathbf{x}-i\omega_{\boldsymbol{\nu}}t}\hat{a}_{\boldsymbol{\nu}} - e^{-i\mathbf{k}_{\boldsymbol{\nu}}\mathbf{x}+i\omega_{\boldsymbol{\nu}}t}\hat{a}^\dagger_{\boldsymbol{\nu}}\right]$$

$$= -\frac{\hbar}{2V} \sum_{\boldsymbol{\mu},\boldsymbol{\nu}\in\mathbb{Z}^3\times\{\pm 1\}} \mathbf{e}_{\boldsymbol{\mu}} \times (\mathbf{k}_{\boldsymbol{\nu}} \times \mathbf{e}_{\boldsymbol{\nu}}) \cdot \int_V d\mathbf{x}$$

$$\left[e^{i(\mathbf{k}_{\boldsymbol{\mu}}+\mathbf{k}_{\boldsymbol{\nu}})\mathbf{x}-i(\omega_{\boldsymbol{\mu}}+\omega_{\boldsymbol{\nu}})t}\hat{a}_{\boldsymbol{\mu}}\hat{a}_{\boldsymbol{\nu}} + e^{-i(\mathbf{k}_{\boldsymbol{\mu}}+\mathbf{k}_{\boldsymbol{\nu}})\mathbf{x}+i(\omega_{\boldsymbol{\mu}}+\omega_{\boldsymbol{\nu}})t}\hat{a}^\dagger_{\boldsymbol{\mu}}\hat{a}^\dagger_{\boldsymbol{\nu}} - e^{i(\mathbf{k}_{\boldsymbol{\nu}}-\mathbf{k}_{\boldsymbol{\mu}})\mathbf{x}-i(\omega_{\boldsymbol{\nu}}-\omega_{\boldsymbol{\mu}})t}\hat{a}^\dagger_{\boldsymbol{\mu}}\hat{a}_{\boldsymbol{\nu}} - e^{i(\mathbf{k}_{\boldsymbol{\mu}}-\mathbf{k}_{\boldsymbol{\nu}})\mathbf{x}-i(\omega_{\boldsymbol{\mu}}-\omega_{\boldsymbol{\nu}})t}\hat{a}_{\boldsymbol{\mu}}\hat{a}^\dagger_{\boldsymbol{\nu}}\right]$$

$$= -\frac{\hbar}{2} \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbb{Z}^3\\ p,q\in\{\pm 1\}}} \mathbf{e}_{(\mathbf{m},p)} \times (\mathbf{k}_{\mathbf{n}} \times \mathbf{e}_{(\mathbf{n},q)})\cdot$$

$$\cdot \left[\delta_{\mathbf{m},-\mathbf{n}}e^{-i(\omega_{\mathbf{m}}+\omega_{\mathbf{n}})t}\hat{a}_{(\mathbf{m},p)}\hat{a}_{(\mathbf{n},q)} + \delta_{\mathbf{m},-\mathbf{n}}e^{i(\omega_{\mathbf{m}}+\omega_{\mathbf{n}})}\hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}^\dagger_{(\mathbf{n},q)}\right.$$

$$\left. - \delta_{\mathbf{m},\mathbf{n}}e^{-i(\omega_{\mathbf{n}}-\omega_{\mathbf{m}})}\hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}_{(\mathbf{n},q)} - \delta_{\mathbf{m},\mathbf{n}}e^{-i(\omega_{\mathbf{m}}-\omega_{\mathbf{n}})t}\hat{a}_{(\mathbf{m},p)}\hat{a}^\dagger_{(\mathbf{n},q)}\right]$$

$$= -\frac{\hbar}{2}\sum_{\mathbf{m}\in\mathbb{Z}^3}\sum_{p,q\in\{\pm 1\}} \mathbf{e}_{(\mathbf{m},p)} \times \left[e^{-i2\omega_{\mathbf{m}}t}(\mathbf{k}_{-\mathbf{m}} \times \underbrace{\mathbf{e}_{(-\mathbf{m},q)}}_{\mathbf{e}_{(\mathbf{m},q)}}) \cdot \hat{a}_{(\mathbf{m},p)}\hat{a}_{(-\mathbf{m},q)} + e^{i2\omega_{\mathbf{m}}t}(\mathbf{k}_{-\mathbf{m}} \times \underbrace{\mathbf{e}_{(-\mathbf{m},q)}}_{\mathbf{e}_{(\mathbf{m},q)}}) \cdot \hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}^\dagger_{(-\mathbf{m},q)}\right.$$

$$\left. - (\mathbf{k}_{\mathbf{m}} \times \mathbf{e}_{(\mathbf{m},q)}) \cdot \hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}_{(\mathbf{m},q)} - (\mathbf{k}_{\mathbf{m}} \times \mathbf{e}_{(\mathbf{m},q)}) \cdot \hat{a}_{(\mathbf{m},p)}\hat{a}^\dagger_{(\mathbf{m},q)}\right]$$

$$\overset{(\clubsuit)}{=} -\frac{\hbar}{2}\sum_{\substack{\mathbf{m}\in\mathbb{Z}^3\\ p\in\{\pm 1\}}} \mathbf{k}_{\mathbf{m}}\left[-e^{-i2\omega_{\mathbf{m}}t}\hat{a}_{(\mathbf{m},p)}\hat{a}_{(-\mathbf{m},p)} - e^{i2\omega_{\mathbf{m}}t}\hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}^\dagger_{(-\mathbf{m},p)} - \hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}_{(\mathbf{m},p)} - \hat{a}_{(\mathbf{m},p)}\hat{a}^\dagger_{(\mathbf{m},p)}\right]$$

wobei verwendet wurde

$$(\clubsuit) \;:\; \mathbf{e}_{(\mathbf{m},p)} \times (\mathbf{k}_{\pm\mathbf{m}} \times \mathbf{e}_{(\mathbf{m},q)}) = \delta_{p,q} \cdot \mathbf{k}_{\pm\mathbf{m}}$$

Wegen

$$\sum_{\mathbf{m}\in\mathbb{Z}^3} \mathbf{k_m} \left[e^{-i2\omega_\mathbf{m} t}\hat{a}_{(\mathbf{m},p)}\hat{a}_{(-\mathbf{m},p)} + e^{i2\omega_\mathbf{m} t}\hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}^\dagger_{(-\mathbf{m},p)}\right]$$

$$= \frac{1}{2}\sum_{\mathbf{m}\in\mathbb{Z}^3} \left[\mathbf{k}_{\boldsymbol{\mu}} e^{-i2\omega_\mathbf{m} t}\hat{a}_{(\mathbf{m},p)}\hat{a}_{(-\mathbf{m},p)} + \mathbf{k}_{\boldsymbol{\mu}} e^{i2\omega_\mathbf{m} t}\hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}^\dagger_{(-\mathbf{m},p)}\right]$$

$$+ \frac{1}{2}\sum_{\mathbf{m}\in\mathbb{Z}^3} \Big[\underbrace{\mathbf{k_m}}_{-\mathbf{k}_{-\mathbf{m}}}\underbrace{e^{-i2\omega_\mathbf{m} t}}_{e^{-i2\omega_{-\mathbf{m}}}}\underbrace{\hat{a}_{(\mathbf{m},p)}\hat{a}_{(-\mathbf{m},p)}}_{\hat{a}_{(-\mathbf{m},p)}\hat{a}_{(\mathbf{m},p)}} + \underbrace{\mathbf{k_m}}_{-\mathbf{k}_{(-\mathbf{m},p)}}\underbrace{e^{i2\omega_\mathbf{m} t}}_{e^{i2\omega_{-\mathbf{m}}}}\underbrace{\hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}^\dagger_{(-\mathbf{m},p)}}_{\hat{a}^\dagger_{(-\mathbf{m},p)}\hat{a}^\dagger_{(\mathbf{m},p)}}\Big]$$

$$\overset{-\mathbf{m}=:\mathbf{n}}{=} \frac{1}{2}\sum_{\mathbf{m}\in\mathbb{Z}^3} \left[\mathbf{k_m} e^{-i2\omega_\mathbf{m} t}\hat{a}_{(\mathbf{m},p)}\hat{a}_{(-\mathbf{m},p)} + \mathbf{k_m} e^{i2\omega_\mathbf{m} t}\hat{a}^\dagger_{(\mathbf{m},p)}\hat{a}^\dagger_{(-\mathbf{m},p)}\right]$$

$$+ \frac{1}{2}\sum_{\mathbf{n}\in\mathbb{Z}^3} \left[-\mathbf{k_n} e^{-i2\omega_\mathbf{n} t}\hat{a}_{(\mathbf{n},p)}\hat{a}_{(-\mathbf{n},p)} - \mathbf{k_n} e^{i2\omega_\mathbf{n} t}\hat{a}^\dagger_{(\mathbf{n},p)}\hat{a}^\dagger_{(-\mathbf{n},p)}\right]$$

$$= 0$$

ist sogar

$$\boxed{\hat{\mathbf{p}}_S = \frac{\hbar}{2}\sum_{\boldsymbol{\mu}\in\mathbb{Z}^3\times\{\pm1\}} \mathbf{k}_{\boldsymbol{\mu}}\cdot\left[\hat{a}^\dagger_{\boldsymbol{\mu}}\hat{a}_{\boldsymbol{\mu}} + \hat{a}_{\boldsymbol{\mu}}\hat{a}^\dagger_{\boldsymbol{\mu}}\right]} \tag{0.3}$$

Somit ergibt sich

$$\hat{\mathbf{p}}_S\left|n_1, n_2, \dots\right\rangle = \frac{\hbar}{2}\sum_{\boldsymbol{\mu}} \mathbf{k}_{\boldsymbol{\mu}}\left[n_{\boldsymbol{\mu}} + (n_{\boldsymbol{\mu}}+1)\right]\left|n_1, n_2, ..\right\rangle$$

wobei

$$\sum_{\boldsymbol{\mu}\in\mathbb{Z}^3\times\{\pm1\}} \mathbf{k}_{\boldsymbol{\mu}} = \frac{1}{2}\sum_{\boldsymbol{\mu}}\left[\mathbf{k}_{\boldsymbol{\mu}} + \underbrace{\mathbf{k}_{\boldsymbol{\mu}}}_{-\mathbf{k}_{-\boldsymbol{\mu}}}\right] \overset{-\boldsymbol{\mu}\to\boldsymbol{\nu}\to\boldsymbol{\mu}}{=} \frac{1}{2}\sum_{\boldsymbol{\mu}}\left[\mathbf{k}_{\boldsymbol{\mu}} - \mathbf{k}_{\boldsymbol{\mu}}\right] = 0$$

also

$$\boxed{\hat{\mathbf{p}}_S\left|n_1, n_2, \dots\right\rangle = \sum_{\boldsymbol{\mu}\in\mathbb{Z}^3\times\{\pm1\}} \hbar n_{\boldsymbol{\mu}}\mathbf{k}_{\boldsymbol{\mu}}\left|n_1, n_2, ..\right\rangle} \tag{0.4}$$

□