Quantenfeldtheorie

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Aufgabe 25

(a) Wegen $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ antivertauscht γ^μ mit allen $\gamma^{\nu\neq\mu}$. Daher:

$$\gamma_5\gamma^\mu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu = (-1)^3 i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^\mu\gamma_5$$

(b) Aus $\gamma^0\gamma^0 = 1,\ \gamma^i\gamma^i = -1$ folgt

$$\gamma_5^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -(-1)^3(-1)^2(-1)^1(-1)^0 \underbrace{\gamma^0\gamma^0}_{1}\underbrace{\gamma^1\gamma^1}_{-1}\underbrace{\gamma^2\gamma^2}_{-1}\underbrace{\gamma^3\gamma^3}_{-1} = 1$$

(c)

$$P_{\mathrm{R,L}}^2 = \frac{1}{4}(1 \pm 2\gamma_5 + \underbrace{\gamma_5^2}_{1}) = \frac{1}{2}(1 \pm \gamma_5) = R_{\mathrm{R,L}}$$

$$P_{\mathrm{R}}P_{\mathrm{L}} = \frac{1}{4}(1 + \cancel{\gamma_5} - \cancel{\gamma_5} - \underbrace{\gamma_5^2}_{1}) = 0$$

$$P_{\mathrm{R}} + P_{\mathrm{L}} \overset{\text{klar}}{=} 1$$

(d) Die Lagrange-Dichte des Dirac-Felder ist gegeben durch

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \quad , \quad \overline{\psi} := \psi^\dagger\gamma^0$$

Dabei gilt

$$\overline{\psi_{\mathrm{L}}} \overset{\text{def.}}{=} (P_{\mathrm{L}}\psi)^\dagger\gamma^0 = \psi^\dagger P_{\mathrm{L}}^\dagger\gamma^0 = \frac{1}{2}\big(\psi^\dagger\gamma^0 - \psi^\dagger \underbrace{\gamma_5\gamma^0}_{-\gamma^0\gamma_5}\big) = \frac{1}{2}\left(\overline{\psi} + \overline{\psi}\gamma_5\right) = \overline{\psi}P_{\mathrm{R}}$$

$$\overline{\psi_{\mathrm{R}}} \overset{\text{analog}}{=} \overline{\psi}P_{\mathrm{L}}$$

Somit kann der kinetische Term geschrieben werden als:

$$i\overline{\psi}\gamma^\mu\partial_\mu\psi = i\overline{\psi}\,\overbrace{(P_L + P_R)}^{1}\,\gamma^\mu\partial_\mu\,\overbrace{(P_L + P_R)}^{1}\,\psi$$

$$= i\overline{\psi}P_L\gamma^\mu\partial_\mu P_L\psi + i\underbrace{\overline{\psi}P_L}_{\overline{\psi_R}}\gamma^\mu\partial_\mu\underbrace{P_R\psi}_{\psi_R} + i\underbrace{\overline{\psi}P_R}_{\overline{\psi_L}}\gamma^\mu\partial_\mu\underbrace{P_L\psi}_{\psi_L} + i\overline{\psi}P_R\gamma^\mu\partial_\mu P_R\psi$$

$$= i\overline{\psi_R}\gamma^\mu\partial_\mu\psi_R + i\overline{\psi_L}\gamma^\mu\partial_\mu\psi_L + i\overline{\psi}\big(P_L\underbrace{\gamma^\mu P_L}_{P_R\gamma^\mu} + \underbrace{P_R\gamma^\mu}_{\gamma^\mu P_L}P_R\big)\partial_\mu\psi$$

$$\overset{P_L P_R = 0}{=} i\overline{\psi_R}\gamma^\mu\partial_\mu\psi_R + i\overline{\psi_L}\gamma^\mu\partial_\mu\psi_L$$

Anderseits ist

$$\overline{\psi_L}\psi_L + \overline{\psi_R}\psi_R = \overline{\psi}\underbrace{P_R P_L}_{0}\psi + \overline{\psi}\underbrace{P_L P_R}_{0}\psi$$

das heißt eine ähnliche Separation ist für den Massenterm nicht möglich!

□

Aufgabe 26

Die Transformation $\psi \mapsto e^{i\alpha}\psi$ erhält offensichtlich den Lagrangian:

$$\mathcal{L}(\psi) \mapsto \mathcal{L}(e^{i\alpha}\psi) = e^{-i\alpha}\overline{\psi}i\partial_\mu\gamma^\mu e^{i\alpha}\psi - \frac{\lambda}{2}\left[\left(e^{-i\alpha}\overline{\psi}e^{i\alpha}\psi\right)^2 - \left(e^{-i\alpha}\overline{\psi}\gamma_5 e^{i\alpha}\psi\right)^2\right] = \mathcal{L}(\psi)$$

was den Noether-Strom

$$j^\nu = \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi)}}_{-i\overline{\psi}\gamma^\nu}(i\alpha)\psi + (i\alpha)\overline{\psi}\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\nu\overline{\psi})}}_{0} = -\alpha\overline{\psi}\gamma^\nu\psi$$

liefert. Wegen

$$\exp\left[-i\alpha\gamma_5\right]\gamma^\mu = \sum_{n=0}^{\infty}\frac{(-i\alpha)^n}{n!}\underbrace{\gamma_5^n\gamma^\mu}_{(-1)^n\gamma^\mu\gamma_5^n} = \gamma^\mu\exp\left[i\alpha\gamma_5\right]$$

führt $\psi \mapsto e^{i\alpha\gamma_5}\psi$ auf den transformierten Lagrangian

$$\mathcal{L}(\psi) \mapsto \mathcal{L}\left(e^{i\alpha\gamma_5}\psi\right) = i\overline{\psi}\underbrace{e^{i\alpha\gamma_5}\gamma^\mu}_{\gamma^\mu e^{-i\alpha\gamma_5}}\partial_\mu e^{i\alpha\gamma_5}\psi - \frac{\lambda}{2}\left[\left(\overline{\psi}\underbrace{e^{i\alpha\gamma_5}e^{i\alpha\gamma_5}}_{e^{2i\alpha\gamma_5}}\psi\right)^2 - \left(\overline{\psi}\underbrace{e^{i\alpha\gamma_5}\gamma_5}_{\gamma^5 e^{i\alpha\gamma_5}}e^{i\alpha\gamma_5}\psi\right)^2\right]$$

$$= i\overline{\psi}\gamma^\mu\partial_\mu\psi - \frac{\lambda}{2}\left[\left(\overline{\psi}e^{2i\alpha\gamma_5}\psi\right)^2 - \left(\overline{\psi}\gamma_5 e^{2i\alpha\gamma_5}\psi\right)^2\right]$$

Die Wirkung des Generators $i\alpha\gamma_5$ auf $\mathcal{L}$ ergibt sich also gemäß

$$\Delta_{i\alpha\gamma_5}\mathcal{L} = \partial_t\mathcal{L}\left(e^{it\alpha\gamma_5}\psi\right)\Big|_{t=0} = \alpha\cdot\left[\partial_\alpha\mathcal{L}\left(e^{i\alpha\gamma_5}\psi\right)\right]_{\alpha=0} = -\lambda\alpha\left[\left(\overline{\psi}\psi\right)\left(\overline{\psi}2i\gamma_5\psi\right) - \left(\overline{\psi}\gamma_5\psi\right)\left(\overline{\psi}\gamma_5 2i\gamma_5\psi\right)\right]$$

$$= -2i\lambda\alpha\underbrace{\left[\left(\overline{\psi}\psi\right)\left(\overline{\psi}\gamma_5\psi\right) - \left(\overline{\psi}\gamma_5\psi\right)\left(\overline{\psi}\psi\right)\right]}_{\left[\overline{\psi}\psi,\overline{\psi}\gamma_5\psi\right]=0} = 0$$

Dementsprechend erhält man den Noether-Strom

$$j^\mu = \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi)}}_{-i\overline{\psi}\gamma^\nu}\underbrace{\Delta_{i\alpha\gamma_5}}_{i\alpha\gamma_5\psi} + \Delta_{i\alpha\gamma_5}\overline{\psi}\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\nu\overline{\psi})}}_{0} = \alpha\overline{\psi}\gamma^\nu\gamma_5\psi$$