Quantenfeldtheorie

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Allgemeine Erinnerungen

Allgemeine Rechenregeln

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \quad , \quad \varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \quad , \quad \varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{km} \tag{1}$$

Aufgabe 21

Vorbetrachtungen

Kontravariante Kommutatoren:

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= g^{\mu\lambda} g^{\nu\alpha} g^{\rho\tau} g^{\sigma\varkappa} [M_{\lambda\alpha}, M_{\tau,\varkappa}] \\
&= -i \left[\delta^{\mu}_{\tau} g^{\nu\alpha} g^{\rho\tau} g^{\sigma\varkappa} M_{\alpha\varkappa} - g^{\mu\lambda}\delta^{\nu}_{\tau} g^{\rho\tau} g^{\sigma\varkappa} M_{\lambda\varkappa} - \delta^{\mu}_{\varkappa} g^{\nu\alpha} g^{\rho\tau} g^{\sigma\varkappa} M_{\alpha\tau} + g^{\mu\lambda} g^{\nu\alpha} g^{\rho\tau}\delta^{\sigma}_{\alpha} M_{\lambda\tau}\right] \\
&= -i \left[g^{\rho\mu} M^{\nu\sigma} - g^{\rho\nu} M^{\mu\sigma} - g^{\sigma\mu} M^{\nu\rho} + g^{\nu\sigma} M^{\mu\rho}\right]
\end{aligned}$$

Speziell für $i, j = 1, 2, 3$ ist

$$M^{ij} = \underbrace{g^{i\mu}}_{-\delta^{i\mu}} \underbrace{g^{j\nu}}_{-\delta^{j\nu}} M_{\mu\nu} = M_{ij} \tag{2}$$

Alternierung von g:

$$\varepsilon_{imn}\varepsilon_{jrs} \underbrace{g^{rm}}_{\substack{-\delta^{rm} \\ r,m=1,2,3}} = -\varepsilon_{imn}\varepsilon_{jms} \overset{(1)}{=} \delta_{is}\delta_{nj} - \delta_{ij}\delta_{ns} \tag{3}$$

Aufgabenausarbeitung

(a) Beginnend mit den Definitionen schreiben wir

$$[J_i, J_j] = \frac{1}{4}\varepsilon_{imn}\varepsilon_{jrs}\left[M^{mn}, M^{rs}\right] = -\frac{i}{4}\varepsilon_{imn}\varepsilon_{jrs}\left[g^{rm}M^{ns} - g^{rn}M^{ms} - g^{sm}M^{nr} + g^{ns}M^{mr}\right]$$

$$= -\frac{i}{4}\varepsilon_{imn}\varepsilon_{jrs}\underbrace{\left[g^{rm}M^{ns} + g^{rm}M^{ns} + g^{rm}M^{ns} + g^{rm}M^{ns}\right]}_{4g^{rm}M^{ns}} \overset{(3)}{=} i[\delta_{ij}\underbrace{\delta_{ns}M^{ns}}_{0} - \delta_{is}\delta_{nj}M^{ns}]$$

$$= iM^{ij} = \frac{i}{2}\left[M^{ij} - M^{ji}\right] = \frac{i}{2}\underbrace{\left[\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}\right]}_{\varepsilon_{ijk}\varepsilon_{krs}}M^{rs} = i\varepsilon_{ijk}J_k$$

$$[J_i, K_j] = \frac{1}{2}\varepsilon_{imn}\left[M^{mn}, M_{j0}\right] = \frac{1}{2}\varepsilon_{imn}g^{m\mu}g^{n\nu}\left[M_{\mu\nu}, M_{j0}\right]$$

$$= -\frac{i}{2}\varepsilon_{imn}g^{m\mu}g^{n\nu}\left[g_{\mu j}M_{\nu 0} - g_{\nu j}M_{\mu 0} - g_{\mu 0}M_{\nu j} + g_{\nu 0}M_{\mu j}\right]$$

$$= -\frac{i}{2}\varepsilon_{imn}\left[\delta^m_j g^{n\nu}M_{\nu 0} - g^{m\mu}\delta^n_j M_{\mu 0} - \underbrace{\delta^m_0}_{\substack{0\\ m=1,2,3}} g^{n\nu}M_{\nu j} + g^{m\mu}\underbrace{\delta^n_0}_{\substack{0\\ m=1,2,3}} M_{\mu j}\right]$$

$$= -\frac{i}{2}\left[\varepsilon_{ijn}g^{n\nu}M_{\nu 0} - \varepsilon_{imj}g^{m\mu}M_{\mu 0}\right] = -i\varepsilon_{ijn}\underbrace{g^{n\nu}M_{\nu 0}}_{\substack{g^{ns}M_{s0}\\ n=1,2,3}} = -i\varepsilon_{ijn}\underbrace{g^{ns}}_{\substack{-\delta^{ns}\\ n=1,2,3}} M_{s0} = i\varepsilon_{ijs}\underbrace{M_{s0}}_{K_{s0}}$$

und schließlich

$$[K_i, K_j] = [M_{i0}, M_{j0}] = -i\left[g_{i,j}\underbrace{M_{00}}_{0} - \underbrace{g_{0j}}_{0}M_{i0} - \underbrace{g_{i0}}_{0}M_{0j} + \underbrace{g_{00}}_{1}M_{ij}\right] = -iM_{ij} \overset{(2)}{=} -iM^{ij} \overset{\substack{\text{analog}\\ \text{zu oben}}}{=} -i\varepsilon_{ijk}J_k$$

(b) Unter Verwendung von (a) schreiben wir

$$[A_i, A_j] = \frac{1}{4}\left\{\underbrace{[J_i, J_j]}_{i\varepsilon_{ijk}J_k} + i\underbrace{[J_i, K_j]}_{i\varepsilon_{ijk}K_k} + i\underbrace{[K_i, J_j]}_{i\varepsilon_{ijk}K_k} - \underbrace{[K_i, K_j]}_{-i\varepsilon_{ijk}J_k}\right\} = \frac{i}{2}\varepsilon_{ijk}\left[J_k + iK_k\right] = i\varepsilon_{ijk}A_k$$

$$[B_i, B_j] = \frac{1}{4}\left\{\underbrace{[J_i, J_j]}_{i\varepsilon_{ijk}J_k} - i\underbrace{[J_i, K_j]}_{i\varepsilon_{ijk}K_k} - i\underbrace{[K_i, J_j]}_{i\varepsilon_{ijk}K_k} - \underbrace{[K_i, K_j]}_{-i\varepsilon_{ijk}J_k}\right\} = \frac{i}{2}\varepsilon_{ijk}\left[J_k - iK_k\right] = i\varepsilon_{ijk}B_k$$

$$[A_i, B_j] = \frac{1}{4}\left\{\underbrace{[J_i, J_j]}_{i\varepsilon_{ijk}J_k} - i\underbrace{[J_i, K_j]}_{i\varepsilon_{ijk}K_k} + i\underbrace{[K_i, J_j]}_{i\varepsilon_{ijk}K_k} + \underbrace{[K_i, K_j]}_{-i\varepsilon_{ijk}J_k}\right\} = 0$$

Aufgabe 22

Für beliebige $a \in \mathrm{SL}(2, \mathbb{C})$ ergibt sich

$$a^T\varepsilon a = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon$$

Somit folgt für beliebige $\xi, \zeta \in \mathbb{C}^2$:

$$(a\xi)\cdot(a\zeta) \overset{\text{def}}{=} (a\xi)^T\varepsilon(a\zeta) = \xi^T\underbrace{(a^T\varepsilon a)}_{\varepsilon}\zeta = \xi\cdot\zeta$$

□

Aufgabe 23

(a) (1) Bekanntlich sind die Pauli-Matrizen spurlos. Wegen $\overline{\sigma}_i\sigma_j = -i\varepsilon_{ijk}\sigma_k$ für $i \neq j \in \{1,2,3\}$ ist

$$\operatorname{trace}(\overline{\sigma}_\mu\sigma_\nu) = \left\{ \begin{array}{ll} 0 & : \mu \neq \nu \\ 2 & : \mu = \nu = 0 \\ -2 & : \mu = \nu \neq 0 \end{array} \right\} = 4\delta_{\mu 0}\delta_{\nu 0} - 2\delta_{\mu\nu}$$

so dass wir schreiben können

$$\frac{1}{2}\operatorname{trace}(\overline{\sigma}^\mu\sigma_\nu) = \frac{g^{\mu\rho}}{2}\operatorname{trace}(\overline{\sigma}_\rho\sigma_\nu) = 2g^{\mu\rho}\delta_{\rho 0}\delta_{\nu 0} - g^{\mu\rho}\delta_{\rho\nu} = 2\underbrace{g^{\mu 0}\delta_{\nu 0}}_{\delta_{\mu 0}\delta_{\nu 0}} - g^{\mu\nu} = \delta^\mu_\nu$$

(2) Mit

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

lässt sich leicht nachprüfen dass

$$\sum_{r=1}^{3}(\sigma_r)_{\alpha\beta}(\sigma_r)_{\gamma\delta} = \left\{ \begin{array}{ll} 1 & : \{(\alpha,\beta),(\gamma,\delta)\} \in \{\{(1,1)\}\,,\ \{(2,2)\}\} \\ -1 & : \{(\alpha,\beta),(\gamma,\delta)\} = \{(1,1),(2,2)\} \\ 2 & : \{(\alpha,\beta),(\gamma,\delta)\} = \{(1,2),(2,1)\} \\ 0 & : \text{sonst} \end{array} \right\} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta} \tag{4}$$

Somit ist

$$(\sigma^\mu)_{\alpha\beta}(\overline{\sigma}_\mu)^{\gamma\delta} = g^{\mu\rho}(\sigma_\rho)_{\alpha\beta}(\overline{\sigma}_\mu)^{\gamma\delta} = \sum_\rho g^{\rho\rho}(\sigma_\rho)_{\alpha,\beta}(\overline{\sigma}_\rho)^{\gamma\delta} = \underbrace{g^{00}}_{1}\underbrace{(\sigma_0)_{\alpha\beta}}_{\delta_{\alpha\beta}}\underbrace{(\overline{\sigma}_0)^{\gamma\delta}}_{\delta_{\gamma\delta}} - \sum_{r=1}^{3}(\sigma_r)_{\alpha\beta}\underbrace{(\overline{\sigma}_r)^{\gamma\delta}}_{(\sigma_r)_{\gamma\delta}}$$

$$= \delta_{\alpha\beta}\delta^{\gamma\delta} + \sum_{r=1}^{3}(\sigma_r)_{\alpha\beta}(\sigma_r)_{\gamma\delta} \overset{(4)}{=} 2\delta^\delta_\alpha\delta^\gamma_\beta$$

(3) Die Fälle $\mu = 0 \neq \nu$ bzw. $\nu = 0 \neq \mu$ sind trivial, ebenso der Fall $\mu = \nu = 0$.
Seien also o.B.d.A. $m := \mu \neq 0,\ n := \nu \neq 0$. Dann ist

$$\sigma_m\underbrace{\overline{\sigma}_n}_{-\sigma_n} + \sigma_n\underbrace{\overline{\sigma}_m}_{-\sigma_m} = -\underbrace{(\sigma_m\sigma_n + \sigma_n\sigma_m)}_{[\sigma_m,\sigma_n]_+ = 2\delta_{mn} = -2g_{mn}} = \underbrace{\overline{\sigma}_m}_{-\sigma_m}\sigma_n + \underbrace{\overline{\sigma}_n}_{-\sigma_n}\sigma_m$$

(b) **Erinnerung:**

$$\Lambda = \exp\left[-\frac{\varepsilon^{\rho\sigma}}{2}M_{\rho\sigma}\right] \quad , \quad a = \exp\left[-\frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}\right]$$

mit

$$(M_{\rho\sigma})^\mu{}_\nu := i\left[\delta^\mu_\rho g_{\sigma\nu} - \delta^\mu_\sigma g_{\rho\nu}\right] \quad , \quad \varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$$

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\sigma_\mu\overline{\sigma}_\nu - \sigma_\nu\overline{\sigma}_\mu\right] \quad , \quad \sigma_\mu := (1,\boldsymbol{\sigma}) \ , \ \overline{\sigma}_\mu := (1,-\boldsymbol{\sigma})$$

Bemerkung: Wegen

$$\sigma^\dagger_{\mu\nu} = -\frac{i}{2}\left[\overline{\sigma}^\dagger_\nu\sigma^\dagger_\mu - \overline{\sigma}^\dagger_\mu\sigma^\dagger_\nu\right] = \frac{i}{2}\left[\overline{\sigma}_\mu\sigma_\nu - \overline{\sigma}_\nu\sigma_\mu\right] =: \overline{\sigma}_{\mu\nu}$$

gilt

$$a^\dagger = \exp\left[\frac{i}{4}\varepsilon^{\mu\nu}\sigma^\dagger_{\mu\nu}\right] = \exp\left[\frac{i}{4}\varepsilon^{\mu\nu}\overline{\sigma}_{\mu\nu}\right]$$

Behauptung:

$$\frac{d}{dt}\sigma_\mu\underbrace{\exp\left[-t\frac{\varepsilon^{\rho\sigma}}{2}M_{\rho\sigma}\right]^\mu{}_\nu}_{\Lambda^\mu{}_\nu(t)}\Bigg|_{t=0} = \frac{d}{dt}\underbrace{\exp\left[-t\frac{i}{4}\varepsilon^{\rho\sigma}\sigma_{\rho\sigma}\right]}_{a(t)}\sigma_\nu\underbrace{\exp\left[t\frac{i}{4}\varepsilon^{\rho\sigma}\overline{\sigma}_{\rho\sigma}\right]}_{a^\dagger(t)}\Bigg|_{t=0}$$

Beweis:

Einerseits ist

$$\frac{d}{dt}\sigma_\mu \Lambda^\mu{}_\nu(t)\Big|_{t=0} = -\frac{i}{2}\sigma_\mu \varepsilon^{\rho\sigma}(M_{\rho\sigma})^\mu{}_\nu = \frac{1}{2}\sigma_\mu \varepsilon^{\rho\sigma}\left[\delta^\mu_\rho g_{\sigma\nu} - \delta^\mu_\sigma g_{\rho\nu}\right]$$

$$= \frac{1}{2}\sigma_\mu \left[\varepsilon^{\mu\sigma} g_{\sigma\nu} - \underbrace{\varepsilon^{\rho\mu}}_{-\varepsilon^{\mu\rho}} g_{\rho\nu}\right] = \sigma_\mu \varepsilon^{\mu\sigma} g_{\sigma\nu} =: (\spadesuit)$$

Anderseits:

$$\frac{d}{dt}a(t)\sigma_\nu a^\dagger(t)\Big|_{t=0} = -\frac{i}{4}\varepsilon^{\rho\sigma}\sigma_{\rho\sigma}\sigma_\nu + \frac{i}{4}\sigma_\nu \varepsilon^{\rho\sigma}\overline{\sigma}_{\rho\sigma} = \frac{1}{8}\varepsilon^{\rho\sigma}\left[\sigma_\rho\overline{\sigma}_\sigma\sigma_\nu - \sigma_\sigma\overline{\sigma}_\rho\sigma_\nu - \sigma_\nu\overline{\sigma}_\rho\sigma_\sigma + \sigma_\nu\overline{\sigma}_\sigma\sigma_\rho\right]$$

$$= \frac{1}{8}\varepsilon^{\rho\sigma}\left[\sigma_\rho\overline{\sigma}_\sigma\sigma_\nu + \sigma_\rho\overline{\sigma}_\sigma\sigma_\nu + \sigma_\nu\overline{\sigma}_\sigma\sigma_\rho + \sigma_\nu\overline{\sigma}_\sigma\sigma_\rho\right] = \frac{1}{4}\varepsilon^{\rho\sigma}\left[\sigma_\rho\overline{\sigma}_\sigma\sigma_\nu + \sigma_\nu\overline{\sigma}_\sigma\sigma_\rho\right]$$

$$\overset{(a)}{=} \frac{1}{4}\varepsilon^{\rho\sigma}\left[\sigma_\rho\left(2g_{\sigma\nu} - \overline{\sigma}_\nu\sigma_\sigma\right) + \left(2g_{\nu\sigma} - \sigma_\sigma\overline{\sigma}_\nu\right)\sigma_\rho\right] = \sigma_\rho\varepsilon^{\rho\sigma}g_{\sigma\nu} - \frac{1}{4}\varepsilon^{\rho\sigma}\left[\sigma_\rho\overline{\sigma}_\nu\sigma_\sigma + \sigma_\sigma\overline{\sigma}_\nu\sigma_\rho\right]$$

$$= \sigma_\rho\varepsilon^{\rho\sigma}g_{\sigma\nu} - \frac{1}{4}\varepsilon^{\rho\sigma}\left[\cancel{\sigma_\rho\overline{\sigma}_\nu\sigma_\sigma} - \cancel{\sigma_\rho\overline{\sigma}_\nu\sigma_\sigma}\right] = (\spadesuit)$$

(Unvollständig)

(c) **Bemerkung 1:** Für beliebige $M \in \mathbb{C}^{2\times 2}$ ist

$$\varepsilon M^T \varepsilon^T = \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} = \begin{cases} \det(M)\cdot M^{-1} & : M \text{ invertierbar} \\ 0 & : \text{sonst} \end{cases}$$

Bemerkung 2: Aus (b) folgt

$$a^{-1}\sigma_\mu (a^\dagger)^{-1}\Lambda^\mu{}_\nu = \sigma_\nu$$

$$\Rightarrow\ a^{-1}\sigma^\rho(a^\dagger)^{-1} = a^{-1}\sigma_\mu(a^\dagger)^{-1}g^{\rho\mu} = a^{-1}\sigma_\mu(a^\dagger)^{-1}\overbrace{\Lambda^\mu{}_\nu\Lambda^\rho{}_\sigma g^{\nu\sigma}}^{g^{\rho\mu}} = \sigma_\nu\Lambda^\rho{}_\sigma g^{\nu\sigma} = \sigma^\sigma\Lambda^\rho{}_\sigma$$

Bemerkung 3: Es ist

$$V_\mu = \xi^\alpha(\sigma_\mu)_{\alpha\beta}\eta^\beta = \varepsilon^{\alpha\beta}\xi_\beta(\sigma_\mu)_{\alpha\beta}\varepsilon^{\beta\gamma}\eta_\gamma = \xi^T\varepsilon^T\sigma_\mu\varepsilon\eta$$

Bemerkung 4: ξ und β transformieren sich gemäß

$$(\xi_\alpha) \mapsto (a_\alpha{}^\beta)(\xi_\beta)\quad,\quad (\eta_\alpha)\mapsto (a^*{}_\alpha{}^\beta)(\eta_\beta)$$

Beweis der Behauptung:

$$V_\mu = \xi^T\varepsilon^T\sigma_\mu\varepsilon\eta \mapsto \xi a^T\varepsilon^T\sigma_\mu\varepsilon a^*\eta = \xi\,\overbrace{\varepsilon^T\varepsilon}^{1}\,a^T\varepsilon^T\sigma_\mu\varepsilon a^*\,\overbrace{\varepsilon^T\varepsilon}^{1}\,\eta \overset{\text{Bem. 1}}{=} \xi\varepsilon^T\det(a)a^{-1}\sigma_\mu\det(\underbrace{a^{*T}}_{a^\dagger})(\underbrace{a^{*T}}_{a^\dagger})^{-1}$$

$$\overset{\det(a)=1}{=} \xi^T\varepsilon^T a^{-1}\sigma_\mu(a^\dagger)^{-1}\varepsilon\eta \overset{\text{Bem. 2}}{=} \xi^T\varepsilon^T\sigma_\sigma\Lambda_\mu{}^\sigma\varepsilon\eta \overset{\text{Bem. 3}}{=} \Lambda_\mu{}^\sigma V_\sigma$$

[1]

References

[1] *Lösungsvorschlag zu Aufgabe 23(c)*, J. Braun
FSU Jena, 2009