Quantenfeldtheorie

FSU Jena - SS 2009

Serie 05 - Lösungen

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15. Juli 2010

Vorbetrachtung

Zeigen dass tatsächlich $E'_{\mathbf{p}} := E_{\mathbf{p}'} = \gamma(E_{\mathbf{p}} - \beta p^3)$ gilt:

$$E_{\mathbf{p}'} = \sqrt{\gamma^2(p^3 - \beta E_{\mathbf{p}})^2 + m^2} = \sqrt{\gamma^2\left[(p^3)^2 - 2\beta E_{\mathbf{p}} p^3 + \beta^2\left((p^3)^2 + m^2\right)\right] + m^2}$$

$$= \sqrt{\gamma^2\left[(p^3)^2 - 2\beta E_{\mathbf{p}} p^3 + \beta^2 (p^3)^2\right] + (\gamma^2\beta^2 + 1)m^2}$$

$$= \sqrt{\gamma^2\left[(p^3)^2 - 2\beta E_{\mathbf{p}} p^3 + \beta^2 (p^3)^2\right] + \gamma^2 m^2}$$

$$= \sqrt{\gamma^2((p^3)^2 + m^2) - 2\gamma^2\beta p^3 E_{\mathbf{p}} + \gamma^2\beta^2 (p^3)^2}$$

$$= \gamma\Bigg[\underbrace{\sqrt{\mathbf{p}^2 + m^2}}_{E_{\mathbf{p}}} - \beta p^3\Bigg]$$

wobei o.B.d.A $\mathbf{p} = (0, 0, p^3)$ gesetzt wurde.

Aufgabe 10

Beginnend mit

$$\left[a(\mathbf{q}), a^\dagger(\mathbf{p})\right] = (2\pi)^d \delta(\mathbf{q} - \mathbf{p})$$

schreiben wir für das Skalarprodukt

$$\langle \mathbf{q}, \mathbf{p}\rangle = \sqrt{2E_{\mathbf{q}}}\sqrt{2E_{\mathbf{p}}}\left\langle a^\dagger(\mathbf{q})\,|0\rangle\,,\ a^\dagger(\mathbf{p})\,|0\rangle\right\rangle = 2\sqrt{E_{\mathbf{q}} E_{\mathbf{p}}}\langle|0\rangle\,,\ \underbrace{a(\mathbf{q}) a^\dagger(\mathbf{p})}_{a^\dagger(\mathbf{p})a(\mathbf{q}) + (2\pi)^d\delta(\mathbf{q}-\mathbf{p})}\ |0\rangle\rangle$$

$$\overset{a(\mathbf{q})|0\rangle = 0}{=} 2(2\pi)^d\sqrt{E_{\mathbf{q}}E_{\mathbf{p}}}\underbrace{\langle 0\ |0\rangle}_{1}\delta(\mathbf{q}-\mathbf{p}) = 2E_{\mathbf{p}}(2\pi)^d\delta(\mathbf{q}-\mathbf{p})$$

Entsprechend

$$\langle \mathbf{q}', \mathbf{p}'\rangle = 2E'_{\mathbf{p}}(2\pi)^d\delta(\mathbf{q}'-\mathbf{p}') = 2E'_{\mathbf{p}}(2\pi)^d\delta({q'}^1 - {p'}^1)\delta({q'}^2 - {p'}^2)\delta\left(\gamma(q^3 - p^3) + \gamma\beta(E_{\mathbf{p}} - E_{\mathbf{q}})\right)$$

$$= 2\gamma(E_{\mathbf{p}} - \beta p^3)(2\pi)^d\delta(\mathbf{q}-\mathbf{p}) \cdot \frac{1}{\left|\gamma - \gamma\beta\frac{\partial E_{\mathbf{q}}}{\partial q^3}\right|} = 2E_{\mathbf{q}}(2\pi)^d\delta(\mathbf{q}-\mathbf{p}) \cdot \frac{\overbrace{E_{\mathbf{p}} - \overbrace{\beta}^{<1} p^3}^{>0}}{|E_{\mathbf{p}} - \beta q^3|} = \langle \mathbf{q}, \mathbf{p}\rangle$$

Aufgabe 11

Schreiben $\Phi(\mathbf{s})$ (bzw. $\Pi(\mathbf{s})$) für $\phi(\mathbf{s})$ (bzw. $\pi(\mathbf{s})$) im Falle eines Operators.

(a) Beginnend mit der Darstellung

$$\Phi(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[a(\mathbf{p}) + a^\dagger(-\mathbf{p})\right] \quad , \quad \Pi(\mathbf{p}) = -i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left[a(\mathbf{p}) - a^\dagger(-\mathbf{p})\right] \tag{1}$$

und

$$[a(\mathbf{p}), a(\mathbf{q})] = 0 \quad , \quad \left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] = \underbrace{(2\pi)^d}_{\varkappa}\delta(\mathbf{p}-\mathbf{q})$$

erhält man

$$[\Phi(\mathbf{p}), \Pi(\mathbf{q})] = \frac{i}{2}\underbrace{\left[a(\mathbf{p}), a^\dagger(-\mathbf{q})\right]}_{\varkappa\delta(\mathbf{p}+\mathbf{q})} + \frac{i}{2}\underbrace{\left[a(\mathbf{q}), a^\dagger(-\mathbf{p})\right]}_{\varkappa\delta(\mathbf{p}+\mathbf{q})} = i\varkappa\cdot\delta(\mathbf{p}+\mathbf{q})$$

In Analogie zum Ortsraum machen wir den Ansatz

$$\Pi(\mathbf{p}) = -i\varkappa\frac{\delta}{\delta\phi(-\mathbf{p})}$$

und gehen damit in den Kommutator ein:

$$\left[\Phi(\mathbf{p}), -i\varkappa\frac{\delta}{\delta\phi(-\pi)}\right]F = -i\varkappa\left\{\Phi(\mathbf{p})\frac{\delta F}{\delta\phi(-\mathbf{q})} - \frac{\delta}{\delta\phi(-\mathbf{q})}\left[\Phi(\mathbf{p})F\right]\right\} = i\varkappa\frac{\delta\phi(\mathbf{p})}{\delta\phi(-\mathbf{q})} = i\varkappa\cdot\delta(\mathbf{p}+\mathbf{q})\,[\Phi(\mathbf{p}), \Pi(\mathbf{q})]$$

(b) Aus Darstellung (1) folgt

$$a(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\omega_{\mathbf{p}}\Phi(\mathbf{p}) + i\Pi(\mathbf{p})\right] \quad , \quad a^\dagger(\mathbf{p}) \overset{\omega_{-\mathbf{p}}=\omega_{\mathbf{p}}}{=} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\omega_{\mathbf{p}}\Phi(-\mathbf{p}) - i\Pi(-\mathbf{p})\right]$$

(c) Beginnend mit

$$a(\mathbf{p})\left|0\right\rangle = 0$$

schreiben wir um:

$$\left[\omega_{\mathbf{p}}\Phi(\mathbf{p}) + i\Pi(\mathbf{p})\right]\left|0\right\rangle = \underbrace{\left[\omega_{\mathbf{p}}\Phi(\mathbf{p}) + \varkappa\frac{\delta}{\delta\phi(-\mathbf{p})}\right]}_{\mathcal{D}_{\mathbf{p}}}\left|0\right\rangle \quad \forall\ \mathbf{p} \tag{2}$$

wobei in diesem Kontext $\left|0\right\rangle$ als Funktional zu betrachten ist:

$$\left|0\right\rangle : (f : \mathbf{p} \mapsto f(\mathbf{p}) \in \mathbb{C}) \mapsto \left\langle f\ \middle|0\right\rangle \in \mathbb{C}$$

(d) Machen den Ansatz

$$\left\langle\phi\ \middle|0\right\rangle = \mathcal{N}\exp\left\{-\frac{1}{2}\int d^d\mathbf{p}\ d^d\mathbf{q}\ \phi(\mathbf{p})K(\mathbf{p},\mathbf{q})\phi(\mathbf{q})\right\}$$

und gehen damit in die DGL (2) ein:

$$\left\langle\phi\right|\mathcal{D}_{\mathbf{s}}\left|0\right\rangle = \omega_{\mathbf{s}}\left\langle\phi\right|\Phi(\mathbf{s})\left|0\right\rangle - \frac{\varkappa}{2}\left\langle\phi\ \middle|0\right\rangle\cdot\frac{\delta}{\delta\phi(-\mathbf{s})}\int d^d\mathbf{p}\ d^d\mathbf{q}\ \phi(\mathbf{p})K(\mathbf{p},\mathbf{q})\phi(\mathbf{q})$$

$$= \omega_{\mathbf{s}}\phi(\mathbf{s})\left\langle\phi\ \middle|0\right\rangle - \frac{\varkappa}{2}\left\langle\phi\ \middle|0\right\rangle\int d^d\mathbf{p}\ d^d\mathbf{q}\ \frac{\delta}{\delta\phi(-\mathbf{s})}\left[\phi(\mathbf{p})K(\mathbf{p},\mathbf{q})\phi(\mathbf{q})\right]$$

$$= \omega_{\mathbf{s}}\phi(\mathbf{s})\left\langle\phi\ \middle|0\right\rangle - \frac{\varkappa}{2}\left\langle\phi\ \middle|0\right\rangle\int d^d\mathbf{p}\ d^d\mathbf{q}\ \left[\delta(\mathbf{p}+\mathbf{s})K(\mathbf{p},\mathbf{q})\phi(\mathbf{q}) + \phi(\mathbf{p})K(\mathbf{p},\mathbf{q})\delta(\mathbf{q}+\mathbf{s})\right]$$

$$= \omega_{\mathbf{s}}\phi(\mathbf{s})\left\langle\phi\ \middle|0\right\rangle - \frac{\varkappa}{2}\left\langle\phi\ \middle|0\right\rangle\left[\int d^d\mathbf{q}\ K(-\mathbf{s},\mathbf{q})\phi(\mathbf{q}) + \int d^d\mathbf{p}\ \phi(\mathbf{p})\mathbb{K}(\mathbf{p},-\mathbf{s})\right]$$

$$= \omega_{\mathbf{s}}\phi(\mathbf{s})\left\langle\phi\ \middle|0\right\rangle - \frac{\varkappa}{2}\underbrace{\left\{\int d^d\mathbf{p}\ \left[K(-\mathbf{s},\mathbf{p}) + K(\mathbf{p},-\mathbf{s})\right]\phi(\mathbf{p})\right\}}_{\overset{!}{=}\omega_{\mathbf{s}}\frac{2}{\varkappa}\phi(\mathbf{s})}\left\langle\phi\ \middle|0\right\rangle \overset{!}{=} 0$$

Setzt man nun

$$K(\mathbf{p}, \mathbf{q}) := \frac{\omega_{\mathbf{p}}}{\varkappa} \delta(\mathbf{q} + \mathbf{p})$$

so ist K der gesuchte Integralkern und dementsprechend

$$\boxed{\langle \phi \,| 0 \rangle = \mathcal{N} \exp \left[-\frac{1}{2\varkappa} \int d^d\mathbf{p} \; \omega_{\mathbf{p}} \phi(\mathbf{p}) \phi(-\mathbf{p}) \right]}$$

Bemerkung: Beachte den Operator-Charakter von $\Phi(\mathbf{p})$:

$$\Phi(\mathbf{p}) : [F : f \mapsto F(f)] \mapsto [\Phi(\mathbf{p})F : f \mapsto f(\mathbf{p}) \cdot F(f)] \quad , \quad f : \{\mathbf{p}\} \to \mathbb{C}$$

bzw. Funktional-Charakter von $\phi(\mathbf{p})$:

$$\phi(\mathbf{p}) : f \mapsto f(\mathbf{p}) \quad , \quad f : \{\mathbf{p}\} \to \mathbb{C}$$