Quantenfeldtheorie

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Serie 04 - Lösungen

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Aufgabe 09

(a) Für lineare Transformation $\Lambda : T_qM \to T_qM$ auf dem Tangentialraum T_qM und Metrik g folgt aus

$$g(\Lambda x, \Lambda x) = g(x,x) \ \ \forall\ x \in T_qM$$

allgemein sogar

$$g(\Lambda x, \Lambda y) = g(x,y) \ \ \forall\ x, y \in T_qM$$

denn

$$\cancel{g(\Lambda x, \Lambda x)} + \cancel{g(\Lambda y, \Lambda y)} + 2g(\Lambda x, \Lambda y) = g(\Lambda(x+y), \Lambda(x+y)) = g(x+y, x+y) = \underbrace{\cancel{g(x,x)}}_{g(\Lambda x, \Lambda x)} + \underbrace{\cancel{g(y,y)}}_{g(\Lambda y, \Lambda y)} + 2g(x,y)$$

Ist nun Λ solch eine Transformation, so gilt insbesondere

$$x^T(\Lambda^T g \Lambda)y = (\Lambda x)g(\Lambda y) = g(\Lambda x, \Lambda y) = g(x,y) = x^T g y \ \ \forall\ x, y \in T_qM$$

(Matrix-Schreibweise) also

$$\boxed{\Lambda^T g \Lambda = g}$$

(b) Schreiben

$$\cancel{g_{k\lambda}} = g_{\mu\nu}\Lambda^\mu_k \Lambda^\nu_\lambda = \cancel{g_{\mu\nu}\delta^\mu_k \delta^\nu_\lambda} + \underbrace{g_{\mu\nu}\delta^\mu_k}_{g_{k\nu}} \varepsilon^\nu{}_\lambda + \underbrace{g_{\mu\nu}\delta^\nu_\lambda}_{g_{\mu\lambda}} \varepsilon^\mu{}_k + g_{\mu\nu}\varepsilon^\mu{}_k \varepsilon^\nu{}_\lambda$$

$$= \varepsilon_{k\lambda} + \varepsilon_{\lambda k} + \mathcal{O}(\varepsilon^2)$$

das heißt

$$\boxed{\varepsilon_{k\lambda} + \varepsilon_{\lambda k} \approx 0}$$

(c) Aus

$$\mathcal{L}(\phi'(x), \partial\phi'(x)) = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi' \big|_x \cdot \partial_\nu \phi' \big|_x - \frac{1}{2}\Omega^2 \phi'^2 \big|_x$$

$$= \frac{1}{2} g^{\mu\nu} \partial_k \phi \big|_{\Lambda x} \cdot \partial_\lambda \phi \big|_{\Lambda x} \cdot \underbrace{\partial_\mu (\Lambda x)^k}_{\Lambda^k_\mu} \underbrace{\partial_\nu (\Lambda x)^\lambda}_{\Lambda^\lambda_\nu} - \frac{1}{2}\Omega^2 \phi'^2 \big|_x$$

$$= \frac{1}{2} g^{\mu\nu} \Lambda^k_\mu \Lambda^\lambda_\nu \cdot \partial_k \phi \big|_{\Lambda x} \cdot \partial_\lambda \phi \big|_{\Lambda x} - \frac{1}{2}\Omega^2 \phi^2 \big|_{\Lambda x}$$

$$\overset{g^{\mu\nu} = g_{\mu\nu}}{=} \frac{1}{2} g^{k\lambda} \partial_k \phi \big|_{\Lambda x} \cdot \partial_\lambda \phi \big|_{\Lambda x} - \frac{1}{2}\Omega^2 \phi^2 \big|_{\Lambda x} = \mathcal{L}\left(\phi\big|_{\Lambda x}, (\partial\phi)\big|_{\Lambda x}\right)$$

und

$$\mathcal{L}\left(\phi\big|_{x+\delta x}, (\partial\phi)\big|_{x+\delta x}\right) = \mathcal{L}\left(\phi\big|_x, (\partial\phi)\big|_x\right) + \frac{\partial\mathcal{L}}{\partial x}\bigg|_x \delta x + \mathcal{O}((\delta x)^2)$$

$$\delta x^\mu := (\Lambda x)^\mu - x^\mu = \varepsilon^\mu{}_k \cdot x^k$$

folgt

$$\mathcal{L}\left(\phi'(x), \partial\phi'(x)\right) = \mathcal{L}(\phi(x), \partial\phi(x)) + \frac{\partial\mathcal{L}}{\partial x^\mu}\bigg|_x \cdot \varepsilon^\mu_k x^k + \mathcal{O}(\varepsilon^2)$$

$$= \mathcal{L}\big|_x + \frac{\partial}{\partial x^\mu}\left[\mathcal{L}\cdot\varepsilon^\mu_k x^k\right]\bigg|_x - \mathcal{L}\big|_x \cdot \varepsilon^\mu_k \underbrace{\partial_\mu x^k}_{\delta^k_\mu} + \mathcal{O}(\varepsilon^2)$$

$$= \mathcal{L}\big|_x + \frac{\partial}{\partial x^\mu}\left[\mathcal{L}\cdot\varepsilon^\mu_k x^k\right]\bigg|_x - \mathcal{L}\big|_x \cdot \underbrace{\varepsilon^\mu_\mu}_{\substack{g^{\mu\nu}\varepsilon_{\mu\nu}\\=0}} + \mathcal{O}(\varepsilon^2)$$

$$= \mathcal{L}\big|_x + \partial_\mu K^\mu + \mathcal{O}(\varepsilon^2)$$

mit

$$K^\mu\big|_x := \mathcal{L}(\phi(x), \partial\phi(x)) \cdot \varepsilon^\mu{}_k x^k$$

Dies entspricht dem Noether-Strom

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial\phi}{\partial x^\nu}\cdot\delta x^\nu - K^\mu = \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial\phi}{\partial x^\nu}\cdot\varepsilon^\nu{}_k - \mathcal{L}\cdot\varepsilon^\mu{}_k\right]\cdot x^k$$

$$= \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left[\partial^\nu\phi\cdot x^k - \partial^k\phi\cdot x^\nu\right]\cdot\varepsilon_{\nu k} - \frac{1}{2}\varepsilon_{\nu k}\left[g^{\mu\nu}\mathcal{L}\cdot x^k - g^{\mu k}\mathcal{L}\cdot x^\nu\right]$$

$$= \frac{1}{2}\varepsilon_{\nu k}\left\{\underbrace{\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}\right]}_{T^{\mu\nu}}\cdot x^k - \underbrace{\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^k\phi - g^{\mu k}\mathcal{L}\right]}_{T^{\mu k}}\cdot x^\nu\right\}$$

$$= \frac{1}{2}\varepsilon_{\nu k}\underbrace{\left[T^{\mu\nu}x^k - T^{\mu k}x^\nu\right]}_{\mathcal{J}^{\mu\nu k}} = \frac{1}{2}\varepsilon_{\nu k}\mathcal{J}^{\mu\nu k}$$

(d) Mit

$$P^\nu = \int d^d\mathbf{x}\ \left[\pi\cdot\partial^\nu\phi - g^{0\nu}\mathcal{L}\right] = \int d^d\mathbf{x}\ \underbrace{T^{0\nu}}_{P^\nu(\mathbf{x})}$$

folgt

$$\int d^d\mathbf{x}\ (\mathbf{x}\times\mathbf{P}(\mathbf{x}))^\nu = \int d^d\mathbf{x}\ x^\lambda P^\nu(\mathbf{x})\cdot\varepsilon_{\lambda\nu k} = \frac{1}{2}\int d^d\mathbf{x}\ \underbrace{\left[x^\lambda T^{0\nu} - x^\nu T^{0\lambda}\right]}_{\mathcal{J}^{0\nu\lambda}}\varepsilon_{\lambda\nu k}$$

$$= -\frac{1}{2}\varepsilon_{k\nu\lambda}\int d^d\mathbf{x}\ \mathcal{J}^{0\nu\lambda} = L^k$$

□