Quantenfeldtheorie

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Aufgabe 07

Schreiben

$$-a_\mu \{\phi(x), P^\mu\} = -a_\mu \int d^dx [\underbrace{\delta_{\phi(z)}\phi(x)}_{\delta(x-z)} \cdot \delta_{\pi(z)}P^\mu - \underbrace{\delta_{\pi(z)}\phi(x)}_{0} \cdot \delta_{\phi(z)}P^\mu]$$

$$= -a_\mu \delta_{\pi(x)}P^\mu = -a_\mu \int d^dz [\underbrace{(\delta_{\pi(x)}\pi(z))}_{\delta(z-x)} \cdot \partial^\mu\phi(z) + \pi(z) \underbrace{\delta_{\pi(x)}\partial^\mu\phi(z)}_{\partial^\mu\delta_{\pi(x)}\phi(z)=0} - g^{\mu 0}\delta_{\pi(x)}\mathcal{L}]$$

$$= -a_\mu\partial^\mu\phi(x) + a^\mu \int d^dz\ g^{\mu 0} \left[\frac{\partial\mathcal{L}}{\partial\phi} \underbrace{\delta_{\pi(x)}\phi(z)}_{0} + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} \underbrace{\delta_{\pi(x)}\partial_\nu\phi(z)}_{\partial_\nu\delta_{\pi(x)}\phi(z)=0}\right]$$

$$= -a_\mu\partial^\mu\phi(x) = -a^\mu\partial_\mu\phi(x) = \delta\phi(x)$$

Aufgabe 08

Kanonische Impulsdichte

Mit

$$\phi = \frac{1}{\sqrt{2}}[\phi_1 + i\phi_2]$$

und

$$\mathcal{L} = \sum_{k=1}^{2} \left\{\frac{1}{2}(\partial_\mu\phi_k)(\partial^\mu\phi_k) - \frac{m^2}{2}\phi_k^2\right\}$$

ergibt sich

$$\pi_k = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_k} = \frac{1}{2}\left[\underbrace{\frac{\partial(\partial_\mu\phi_k)}{\partial(\partial_0\phi_k)}}_{\delta_{0\mu}}(\partial_\nu\phi_k)g^{\mu\nu} + (\partial_\mu\phi_k)\underbrace{\frac{\partial(\partial_\nu\phi_k)}{\partial(\partial_0\phi_k)}}_{\delta_{0\nu}} g^{\mu\nu}\right] = \frac{1}{2}\left[(\partial_\nu\phi_k)g^{0\nu} + (\partial_\mu\phi_k)g^{\mu 0}\right]$$

$$= (\partial_\nu\phi_k)g^{0\nu} = \partial^0\phi_k$$

und somit

$$\partial^0\phi = \frac{1}{\sqrt{2}}[\pi_1 + i\pi_2] =: \pi \quad .$$

Hamilton-Dichte

Schreiben

$$\mathcal{H} = \sum_{k=1}^{2} \pi_k \dot{\phi}_k - \mathcal{L} = \sum_{k=1}^{2} \left[\frac{1}{2}\pi_k^2 + \frac{1}{2}\nabla\phi_k \cdot \nabla\phi_k + \frac{m^2}{2}\phi_k^2 \right]$$

$$= \frac{1}{2}\left(\pi_1 - i\pi_2\right)\left(\pi_1 + i\pi_2\right) + \frac{1}{2}\left(\nabla\phi_1 - i\nabla\phi_2\right)\left(\nabla\phi_1 + i\nabla\phi_2\right) + \frac{m^2}{2}\left(\phi_1 - i\phi_2\right)\left(\phi_1 + i\phi_2\right)$$

$$= |\pi|^2 + \nabla\phi^* \cdot \nabla\phi + \frac{m^2}{2}|\phi|^2$$

Spektrale Darstellung der Hamilton-Funktion

Nennen $\mathcal{F}(f)(k) =: \tilde{f}(k)$ die Fourier-Transformation von f und schreiben

$$H = \int d^dx\ \mathcal{H}(\phi(x), \pi(x)) = \int d^dx \left[|\pi|^2 + \sum_{m=1}^{d} \partial_m\phi^* \cdot \partial_m\phi + m^2|\phi|^2 \right]$$

$$\overset{\text{Parseval}}{=} \int d^dk \left[|\tilde{\pi}(k)|^2 + m^2|\tilde{\phi}(k)|^2 \right] + \frac{1}{(2\pi)^n} \sum_{m=1}^{d} \int d^dx \int d^dk\ \underbrace{\mathcal{F}(\partial_m\phi^*)(k)}_{\substack{ik_m\mathcal{F}(\phi^*)(k) \\ =ik_m(\mathcal{F}(\phi))^*(-k)}} \cdot e^{ikx} \cdot \int d^dk'\ \underbrace{\mathcal{F}(\partial_m\phi)}_{ik'_m\mathcal{F}(\phi)}(k') \cdot e^{ik'x}$$

$$= \int d^dk \left[|\tilde{\pi}(k)|^2 + m^2|\tilde{\phi}(k)|^2 \right] - \frac{1}{(2\pi)^n} \sum_{m=1}^{d} \int\int d^dk\ d^dk'\ k_m k'_m \cdot (\mathcal{F}(\phi))^*(-k) \cdot \mathcal{F}(\phi)(k') \cdot \underbrace{\int d^dx\ e^{i(k+k')x}}_{(2\pi)^n\delta(k+k')}$$

$$= \int d^dk \left[|\tilde{\pi}(k)|^2 + m^2|\tilde{\phi}(k)|^2 \right] + \sum_{m=1}^{d} \int d^dk'\ {k'_m}^2 (\mathcal{F}(\phi))^*(k')\mathcal{F}(\phi)(k')$$

$$= \int d^dk \left[|\mathcal{F}(\pi)(k)|^2 + \|k\|^2 |\mathcal{F}(\phi)(k)|^2 + m^2 |\mathcal{F}(\phi)(k)|^2 \right]$$

Quantisierung der Feldoperatoren

Beginnend mit

$$[a_k(\mathbf{p}), a_j(\mathbf{q})] = \left[a_k^\dagger(\mathbf{p}), a_j^\dagger(\mathbf{q})\right] = 0 \quad , \quad \left[a_k(\mathbf{p}), a_j^\dagger(\mathbf{q})\right] = \underbrace{(2\pi)^d}_{\varkappa} \delta_{kj}\delta(\mathbf{p} - \mathbf{q})$$

schreiben wir

$$[a(\mathbf{p}), a(\mathbf{q})] = \frac{1}{2}\left[a_1(\mathbf{p}) + ia_2(\mathbf{p}), a_2(\mathbf{q}) + ia_2(\mathbf{q})\right] = \underbrace{[a_1(\mathbf{p}), a_1(\mathbf{q})]}_{0} + i\underbrace{[a_1(\mathbf{p}), a_2(\mathbf{q})]}_{0} + i\underbrace{[a_2(\mathbf{p}), a_1(\mathbf{q})]}_{0} - \underbrace{[a_2(\mathbf{p}), a_2(\mathbf{q})]}_{0} = 0$$

$$[b(\mathbf{p}), b(\mathbf{q})] \overset{\text{Analog}}{=} 0 \quad , \quad [a(\mathbf{p}), b(\mathbf{q})] = -\left[a^\dagger(\mathbf{p}), b^\dagger(\mathbf{q})\right] \overset{\text{Analog}}{=} 0$$

$$\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] = \frac{1}{2}\underbrace{\left[a_1(\mathbf{p}), a_1^\dagger(\mathbf{q})\right]}_{\varkappa\delta(\mathbf{p}-\mathbf{q})} + \frac{1}{2}\underbrace{\left[a_2(\mathbf{p}), a_2^\dagger(\mathbf{q})\right]}_{\varkappa\delta(\mathbf{p}-\mathbf{q})} = \varkappa\delta(\mathbf{p} - \mathbf{q}) \quad , \quad \left[b(\mathbf{p}), b^\dagger(\mathbf{q})\right] \overset{\text{Analog}}{=} \varkappa\delta(\mathbf{p} - \mathbf{q})$$

$$\left[a(\mathbf{p}), b^\dagger(\mathbf{q})\right] = \underbrace{\frac{1}{2}\left[a_1(\mathbf{p}), a_1^\dagger(\mathbf{q})\right]}_{\varkappa\delta(\mathbf{p}-\mathbf{q})} - \frac{1}{2}\underbrace{\left[a_2(\mathbf{p}), a_2^\dagger(\mathbf{q})\right]}_{\varkappa\delta(\mathbf{p}-\mathbf{q})} = 0$$

Mit

$$a_1(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[a(\mathbf{p}) + b(\mathbf{p})\right] \quad , \quad a_1^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[a^\dagger(\mathbf{p}) + b^\dagger(\mathbf{p})\right]$$

$$a_2(\mathbf{p}) = \frac{i}{\sqrt{2}}\left[b(\mathbf{p}) - a(\mathbf{p})\right] \quad , \quad a_2^\dagger(\mathbf{p}) = \frac{i}{\sqrt{2}}\left[a^\dagger(\mathbf{p}) - b^\dagger(\mathbf{q})\right]$$

und

$$\phi_k(\mathbf{p}) = \frac{1}{\sqrt{2\omega_\mathbf{p}}}\left[a_k(\mathbf{p}) + a_k^\dagger(-\mathbf{p})\right] \quad , \quad \pi_k(\mathbf{p}) = -i\sqrt{\frac{\omega_\mathbf{p}}{2}}\left[a_k(\mathbf{p}) - a_k^\dagger(-\mathbf{p})\right]$$

ergibt sich ferner

$$\phi(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[\phi_1(\mathbf{p}) + i\phi_2(\mathbf{p})\right] = \frac{1}{\sqrt{2\omega_\mathbf{p}}}\left[a(\mathbf{p}) + b^\dagger(-\mathbf{p})\right]$$

$$\pi(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[\pi_1(\mathbf{p}) + i\pi_2(\mathbf{p})\right] = i\sqrt{\frac{\omega_\mathbf{p}}{2}}\left[b^\dagger(-\mathbf{p}) - a(\mathbf{p})\right]$$

Darstellung des Hamilton-Operators

Eingesetzt in obiger Darstellung für H ergibt

$$H = \frac{1}{\alpha}\int d^d\mathbf{p}\ \left[\pi(\mathbf{p})\pi^\dagger(\mathbf{p}) + \omega_\mathbf{p}^2 \cdot \phi(\mathbf{p})\phi^\dagger(\mathbf{p})\right]$$

$$= \frac{1}{\alpha}\int d^d\mathbf{p}\ \omega_\mathbf{p} \cdot [\ \underbrace{a(\mathbf{p})a^\dagger(\mathbf{p})}_{\varkappa(\delta(0)+a^\dagger(\mathbf{p})a(\mathbf{p})} + b^\dagger(-\mathbf{p})b(-\mathbf{p})]$$

$$= \frac{1}{(2\pi)^d}\int d^d\mathbf{p}\ \omega_\mathbf{p} \cdot \left[a^\dagger(\mathbf{p})a(\mathbf{p}) + b^\dagger(\mathbf{p})b(\mathbf{p})\right] + \underbrace{\int d^d\mathbf{p}\ \omega_\mathbf{p}\delta(0)}_{\text{N.P.E.}}$$

Noether-Ladung

Schreiben Q um:

$$Q = i\int d^d\mathbf{x}\ \left[\phi^\dagger(\mathbf{x})\pi(\mathbf{x}) - \phi(\mathbf{x})\pi^\dagger(\mathbf{x})\right]$$

$$= \frac{i}{\varkappa^2}\int\int\int d^d\mathbf{x}\ d^d\mathbf{p}\ d^d\mathbf{q}\ \left[\phi^\dagger(\mathbf{p})\pi(\mathbf{q})e^{i\mathbf{x}(\mathbf{q}-\mathbf{p})} - \phi(\mathbf{p})\pi^\dagger(\mathbf{q})e^{i\mathbf{x}(\mathbf{p}-\mathbf{q})}\right]$$

$$= \frac{i}{\varkappa^2}\int d^d\mathbf{p}\ d^d\mathbf{q}\ \left[\phi^\dagger(\mathbf{p})\pi(\mathbf{q})\underbrace{\int d^d\mathbf{x}\ e^{i\mathbf{x}(\mathbf{q}-\mathbf{p})}}_{\varkappa\delta(\mathbf{q}-\mathbf{p})} - \phi(\mathbf{p})\pi^\dagger(\mathbf{q})\underbrace{\int d^d\mathbf{x}\ e^{i\mathbf{x}(\mathbf{p}-\mathbf{q})}}_{\varkappa\delta(\mathbf{p}-\mathbf{q})}\right]$$

$$= \frac{i}{(2\pi)^d}\int d^d\mathbf{p}\left[\phi^\dagger(\mathbf{p})\pi(\mathbf{p}) - \phi(\mathbf{p})\pi^\dagger(\mathbf{p})\right]$$

Durch obige Darstellungen von ϕ und π ferner

$$Q = \frac{i}{\varkappa}\int d^d\mathbf{p}\ \frac{i}{2}\left\{\ \left[a^\dagger(\mathbf{p}) + b(-\mathbf{p})\right]\cdot\left[b^\dagger(-\mathbf{p}) - a(\mathbf{p})\right] + \left[a(\mathbf{p}) + b^\dagger(-\mathbf{p})\right]\cdot\left[b(-\mathbf{p}) - a^\dagger(\mathbf{p})\right]\ \right\}$$

$$= \frac{1}{2\varkappa}\int d^d\mathbf{p}\ \left[\underbrace{a(\mathbf{p})a^\dagger(\mathbf{p})}_{a^\dagger a+\varkappa\delta(0)} + a^\dagger(\mathbf{p})a(\mathbf{p}) - b^\dagger(-\mathbf{p})b(-\mathbf{p}) - \underbrace{b(-\mathbf{p})b^\dagger(-\mathbf{p})}_{b^\dagger b+\varkappa\delta(0)}\right]$$

$$= \frac{1}{\varkappa}\int d^d\mathbf{p}\ \left[a^\dagger(\mathbf{p})a(\mathbf{p}) - b^\dagger(-\mathbf{p})b(-\mathbf{p})\right] = \frac{1}{(2\pi)^d}\int d^d\mathbf{p}\ \left[a^\dagger(\mathbf{p})a(\mathbf{p}) - b^\dagger(\mathbf{p})b(\mathbf{p})\right]$$