Introduction to the Worldline Formalism
- Seminar Report -

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Contents

1 Conventions

We use the metric signature $(+,-,..,-)$, denoting 4-vectors $\mathbf{x}$ with **bold** symbols. We denote by $\mathbf{xy}$ the Euclidean scalar product of the 4-vectors $\mathbf{x},\mathbf{y}$, by $g(\mathbf{x},\mathbf{y}) = x^0 y^0 - \sum_{j=1}^{3} x^j y^j$ their Minkowskian one. Greek indices run (unless stated otherwise) from 0 to 3, latin ones only from 1 to 3. Components of linear functionals are written down, components of vectors up. Operators are symbolized with a hat "$\hat{}$".

We use the isometric Fourier-transform and natural units $\hbar = 1 = c$.

2 Introduction

The path integral formalism[1, 2, 9, 18] developed by Feynman[3] in the first half of the 20th century, has come to be a successful complement to the non-relativistic quantum mechanical formalism developed by Schrödinger and Heisenberg. For a wide class of Hamiltonians $\hat{H} = \hat{H}(\hat{\mathbf{x}},\hat{\mathbf{p}})$, the Green's function for the Schrödinger equation, that is, the transition amplitude

$$\langle \mathbf{x}_{\text{out}}, t_{\text{out}} \,|\mathbf{x}_{\text{in}}, t_{\text{in}}\rangle := \langle \mathbf{x}_{\text{out}}|\, e^{-i(t_{\text{out}}-t_{\text{in}})\hat{H}}\, |\mathbf{x}_{\text{in}}\rangle \quad , \tag{2.1}$$

can be expressed by means of a so called path integral

$$\langle \mathbf{x}_{\text{out}}, t_{\text{out}} \,|\mathbf{x}_{\text{in}}, t_{\text{in}}\rangle = \int\limits_{(t_{\text{in}},\mathbf{x}_{\text{in}})}^{(t_{\text{out}},\mathbf{x}_{\text{out}})} D\mathbf{x}\; D\mathbf{p}\; \exp\left[iS[\mathbf{x},\mathbf{p}] \right] \quad . \tag{2.2}$$

Expression (2.2) represents an interference of all possible trajectories $\mathbf{x}(t), \mathbf{p}(t)$ from point $\mathbf{x}_{\text{in}}$ to point $\mathbf{x}_{\text{out}}$ during time $t_{\text{in}} - t_{\text{out}}$, each with the action

$$S[\mathbf{x},\mathbf{p}] := \int\limits_{t_{\text{in}}}^{t_{\text{out}}} \left[\dot{\mathbf{x}}\mathbf{p} - H(\mathbf{x},\mathbf{p}) \right] \, dt \tag{2.3}$$

as a phase. For many special cases of Hamiltonians, such as the ones quadratic in the momentum, the $\mathbf{p}$-integral can readily be evaluated[1, 18] to yield a path integral

$$\langle \mathbf{x}_{\text{out}}, t_{\text{out}} \,|\mathbf{x}_{\text{in}}, t_{\text{in}}\rangle = \int\limits_{(t_{\text{in}},\mathbf{x}_{\text{in}})}^{(t_{\text{out}},\mathbf{x}_{\text{out}})} D\mathbf{x}\; \exp\left[i \int\limits_{t_{\text{in}}}^{t_{\text{out}}} \mathcal{L}(\mathbf{x},\dot{\mathbf{x}}) \, dt \right] \tag{2.4}$$

solely over configuration-space trajectories, with $\mathcal{L}$ as the Lagrangian corresponding to the classical version of $\hat{H}$. The apparent simplicity of the path integral formalism and its similarity to classical mechanical variational methods, makes its generalization to functional integrals over fields a promising alternative to the 2nd quantization description used in quantum field theory and in particular the calculation of N-point correlators[10].

In the 2nd half of the 20th century, a new approach to the calculation of generating functionals and Green's functions in field theory appeared, initiated by Fock, Feynman[8] and Schwinger[16]. This so called *worldline formalism* can be used to express quantum field theoretic objects, such as Green's and 1PI-N-point functions, using solely one-parameter path integrals.

Though simple problems such as the ones handled below, can generally be solved much easier using 2nd quantization methods and Wick contractions, the worldline formalism shows its real strength in more complex situations. In particular, *worldline numerics* has lately enjoyed a wide span of applications ranging from the Casimir effect[20, 21] to pair production in inhomogeneous fields[22, 23]. Furthermore, the intuitive interpretation of worldline integrals, allows for a completely different insight into the whole theory, just as path integrals do in quantum mechanics.

Finally, the worldline formalism shows great analogy to similar results in string theory, where string scattering amplitudes reduce to one-parameter path integral expressions. In fact, as string theory reduces to quantum field theory in the infinite tension limit, string scattering amplitudes reduce to quantum field theoretic ones, thus *inspiring* the development of a worldline formalism eventually without reference to string theory[4, 11, 19].

The following article aims at introducing this formalism and illustrating its applications to various problems usually handled by means of 2nd quantization and Feynman diagrams. It is mainly based on [4],[8],[11],[12] and [15].

3 Green's function for the KG operator

In 1950, Feynman[8] introduced a one-parameter path-integral expression for the Green's function of the Klein-Gordon (KG) operator. We shall in the following briefly illustrate his ideas, mainly as a motivation for what follows thereafter. Consider the KG equation

$$\left(\boxdot + \sigma + m^2\right) \varphi(\mathbf{x}) = 0 \quad , \quad \boxdot := (\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu) \tag{3.1}$$

for a spinless, complex field φ of charge e, interacting with a background gauge field $\mathbf{A}$ and a source $\sigma(\mathbf{x})$. Consider now the differential equation

$$i\partial_u \zeta(\mathbf{x}, u) = \hat{H}\zeta(\mathbf{x}, u) \quad , \quad \hat{H} := \boxdot + \sigma \tag{3.2}$$

for the field $\zeta(\mathbf{x}, u)$, in some 5th parameter u. Since $\hat{H}$ does not depend on u, solutions of (3.2) can be written as combinations of the fundamental solutions

$$\zeta(\mathbf{x}, u) = e^{-i\mathcal{E}u}\varphi(\mathbf{x}) \quad , \tag{3.3}$$

with φ solving the eigenvalue problem $\hat{H}\varphi \overset{!}{=} \mathcal{E}\varphi$. In particular, φ solves (3.1) if, and only if, $\mathcal{E} = -m^2$. Thus, for any solution $\zeta(\mathbf{x}, u)$ of (3.2) vanishing for $u \to \pm\infty$, the field

$$\varphi(\mathbf{x}) := \int_{\mathbb{R}} du \; e^{-im^2 u}\zeta(\mathbf{x}, u) \tag{3.4}$$

solves (3.1). Now suppose $K(\mathbf{x}, u; \mathbf{x}', 0)$ is the retarded Green's function for (3.2), that is, solves (3.2) and satisfies

$$K(\mathbf{x}, 0; \mathbf{x}', 0) = \delta(\mathbf{x} - \mathbf{x}') \quad , \quad K(\mathbf{x}, u; \mathbf{x}', 0) = 0 \text{ for } u < 0 \quad . \tag{3.5}$$

Then it is straightforward to show that the 2-point function

$$G(\mathbf{x}, \mathbf{x}') := i \int_0^\infty e^{-im^2 u} K(\mathbf{x}, u; \mathbf{x}', 0) \; du \tag{3.6}$$

is a Green's function for the Klein-Gordon operator $(\boxdot + \sigma + m^2)$. Its domain is the whole $\mathbb{R}^4$ and its boundary condition that it vanishes for $\|\mathbf{x} - \mathbf{x}'\| \to \infty$. From the theory of path integrals, we know that the transition amplitude for the *Schrödinger equation* (3.2) is given by

$$K(\mathbf{x}, u; \mathbf{x}', 0) = \langle \mathbf{x} | \, e^{-iu\hat{H}} \, | \mathbf{x}' \rangle = \int_{(0,\mathbf{x}')}^{(u,\mathbf{x})} D\mathbf{y} \; \exp\left[i \int_0^u \mathcal{L}(\mathbf{y}, \dot{\mathbf{y}}) \; d\tau \right] \quad , \quad u \geq 0 \quad , \tag{3.7}$$

with

$$\mathcal{L}(\mathbf{y}, \dot{\mathbf{y}}) = -\frac{1}{4}\dot{y}^\mu \dot{y}_\mu + e\dot{y}^\mu A_\mu(\mathbf{y}) - \sigma(\mathbf{y}) \tag{3.8}$$

as the Lagrangian corresponding to the *Hamiltonian*

$$H = -p^\mu p_\mu - e^2 A^\mu A_\mu + 2eA^\mu p_\mu + \sigma \quad . \tag{3.9}$$

The final expression for the Green's function thus becomes

$$\boxed{G(\mathbf{x}, \mathbf{x}') := i \int_0^\infty du \; e^{-im^2 u} \int_{(0,\mathbf{x}')}^{(u,\mathbf{x})} D\mathbf{y} \; \exp\left[-i \int_0^u \left[\frac{1}{4}\dot{y}^\mu \dot{y}_\mu - e\dot{y}^\mu A_\mu + \sigma \right] \; d\tau \right] \quad .} \tag{3.10}$$

4 Applications in QED

Having seen how the worldline formalism can be readily used to express Green's functions, we shall now turn to the problem of solving the theory for a spin-$\frac{1}{2}$ field, of charge e and mass m, interacting with a background gauge field $\mathbf{A}$ and described by the Lagrangian

$$\mathcal{L} = i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi \quad , \quad D_\mu := (\partial_\mu - ieA_\mu) \quad . \tag{4.1}$$

We shall not include the kinetic energy term of the gauge field, as we are eventually interested in objects, such as the polarization tensor, arising from the interaction of the spinor with the gauge field.

The theory is considered to be solved, if its S-matrix, or equivalently, its N-point correlators are known. The calculation of the latter reduces to the calculation of the so called *one-particle irreducible N-point functions*, the functional derivatives[1] of the *effective action* $\Gamma[\psi, \overline{\psi}, \mathbf{A}]$, a functional on fields[10, 17].

We perform a so called *Wick rotation* to imaginary times $x^0 \to -ix^0$, change the fields $A_0 \to iA_0$ and define the *Euclidean* γ-matrices

$$\widetilde{\gamma}^0 := \gamma^0 \quad , \quad \widetilde{\gamma}^j := -i\gamma^j \quad . \tag{4.2}$$

All other coordinates and field components are kept unchanged. We obtain in these new coordinates and fields the representation for the effective action:

$$\begin{aligned} \Gamma[\psi, \overline{\psi}, \mathbf{A}] &= -i \ln \underbrace{\int D\psi' \; D\overline{\psi}\,'}_{\substack{\text{Grassmann} \\ \text{path integral}}} \; \exp\left[iS[\psi, \overline{\psi}] + \int d\mathbf{x} \; \overline{\psi}\,'(-i\widetilde{\gamma}^\mu D_\mu - im)\psi' \right] \\ &= S[\psi, \overline{\psi}, \mathbf{A}] \underbrace{-i \ln \det\left[-i\widetilde{\gamma}^\mu D_\mu - im\right]}_{\Gamma^{11}} + \text{const} \end{aligned} \tag{4.3}$$

The double path integral used above is performed over Grassmann valued paths, with $\psi', \overline{\psi}'$ as 4-dimensional, real Grassmann-valued vectors[2]. The constant in (4.3) is merely dependent on the exact field measure used.

4.1 The one-loop correction to the effective action

Our goal is now, to find a path-integral expression for the 2nd term Γ^{11} in (4.3), the so called *one-loop correction*. Note that Γ^{11} does only depend on the gauge field $\mathbf{A}$. Using Frullani's integral identity[3]

$$\ln \det A = \operatorname{Tr} \ln A = -\int_0^\infty \frac{dT}{T} \operatorname{Tr} e^{-TA} \tag{4.4}$$

for positive operators A and the fact that[15]

$$\begin{aligned} \operatorname{Det}\left[-i\widetilde{\gamma}^\mu D_\mu - im\right] &= \operatorname{Det}^{\frac{1}{2}} \left[(\widetilde{\gamma}^\mu \hat{p}_\mu - e\widetilde{\gamma}^\mu A_\mu)^2 + m^2 \right] \\ &= \operatorname{Det}^{\frac{1}{2}} \underbrace{\left[(\hat{\mathbf{p}} - e\mathbf{A})^2 + \frac{ie}{4} \left[\widetilde{\gamma}^\mu, \widetilde{\gamma}^\nu\right] F_{\mu\nu} + m^2 \right]}_{\mathfrak{D}(\hat{\mathbf{p}}, \hat{\mathbf{A}}, \widetilde{\gamma}^\mu \widetilde{\gamma}^\nu)} = \operatorname{Det}^{\frac{1}{2}} \mathfrak{D} \quad , \end{aligned} \tag{4.5}$$

we obtain the expression

$$\Gamma^{11}[\mathbf{A}] = \frac{i}{2} \int_0^\infty \frac{dT}{T} \operatorname{Tr} e^{-T\mathfrak{D}[\hat{\mathbf{p}}, \mathbf{A}, (\widetilde{\gamma}_\mu \widetilde{\gamma}_\nu)]} \quad . \tag{4.6}$$

[1] Evaluated at the vacuum expectation value $\mathbf{A}(\mathbf{x}) := \langle 0 | \hat{\mathbf{A}}(\mathbf{x}) | 0 \rangle$, which happens in this theory to be $\mathbf{A} \equiv 0$.
[2] See [1, 7, 18] on Grassmann numbers and Grassmann integrals.
[3] Up to an additive, universal constant, which shall be ignored from now on.

Thus, the calculation of $\Gamma^{11}[\mathbf{A}]$ reduces to the calculation of $\operatorname{Tr} e^{-T\mathfrak{D}}$. Due to the higher dimension of the operator $\mathfrak{D}$ resulting from the γ-matrices $\widetilde{\gamma}^{\mu}$, this trace can not be taken simply by integrating over all diagonal elements $\langle \mathbf{x}| e^{-T\mathfrak{D}} |\mathbf{x}\rangle$. Instead, one also has to consider the trace over the Hilbert space on which the γ-matrices act, which is isomorphic to $\mathbb{C}^4$.

For this reason, we introduce the fermionic creation and annihilation operators[4]

$$a_1^{\pm} := \frac{1}{2}(\widetilde{\gamma}^1 \pm i\widetilde{\gamma}^3) \quad , \quad a_2^{\pm} := \frac{1}{2}(\widetilde{\gamma}^2 \pm i\widetilde{\gamma}^0) \quad , \tag{4.7}$$

satisfying the known fermionic anticommutation relations, and the real Grassmann variables[4] $\vartheta^1, \vartheta^2, \overline{\vartheta}^1, \overline{\vartheta}^2$, anticommuting with the $a_{1,2}^{\pm}$ and commuting with the vacuum $|0\rangle$. The operators $a_{1,2}^{\pm}$ generate the Fock-space $\mathcal{H}_a \cong \mathbb{C}^4$ on which the $\widetilde{\gamma}$-matrices operate, spanned by the induced states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, which could be used to define a trace on $\mathcal{H}_a$. Unfortunately, these prove to be somewhat cumbersome when it comes to evaluating $\operatorname{Tr} e^{-T\mathfrak{D}}$. A more suitable basis is given by the coherent states[13, 14]

$$|\boldsymbol{\vartheta}\rangle := \exp\left[-\vartheta^1 a_1^+ - \vartheta^2 a_2^+\right] |0\rangle \qquad \left|\overline{\boldsymbol{\vartheta}}\right\rangle := i(\overline{\vartheta}^1 - a_1^+)(\overline{\vartheta}^2 - a_2^+)\,|0\rangle \quad ,$$

$$\langle\boldsymbol{\vartheta}| := i\,\langle 0|\,(\vartheta^1 - a_1^-)(\vartheta^2 - a_2^-) \qquad \left\langle\overline{\boldsymbol{\vartheta}}\right| := \langle 0|\exp\left[-a_1^-\overline{\vartheta}^1 - a_2^-\overline{\vartheta}^2\right] \quad , \tag{4.8}$$

satisfying[4]

$$\mathbb{1}_{\mathcal{H}_a} = i\int d^2\boldsymbol{\vartheta}\;\; |\boldsymbol{\vartheta}\rangle\langle\boldsymbol{\vartheta}| = -i\int d^2\overline{\boldsymbol{\vartheta}}\;\; \left|\overline{\boldsymbol{\vartheta}}\right\rangle\left\langle\overline{\boldsymbol{\vartheta}}\right| \tag{4.9}$$

and for any operator U on $\mathcal{H}_a$:

$$\operatorname{Tr} U = i\int d^2\boldsymbol{\vartheta}\;\; \langle -\boldsymbol{\vartheta}|\, U\, |\boldsymbol{\vartheta}\rangle \quad . \tag{4.10}$$

Using these states, the 1-loop correction to the effective action (4.6) can be written as

$$\Gamma^{11}[\mathbf{A}] = -\frac{1}{2}\int_0^{\infty}\frac{dT}{T}\int d^4\mathbf{x}\; d\vartheta^1\; d\vartheta^2\;\; \langle \mathbf{x}, -\boldsymbol{\vartheta}|\, e^{-T\mathfrak{D}[\hat{\mathbf{p}},\mathbf{A},(\widetilde{\gamma}_\mu\widetilde{\gamma}_\nu)]}\,|\mathbf{x},\boldsymbol{\vartheta}\rangle \quad . \tag{4.11}$$

By performing an N-fold slicing of the interval $[0,T]$ and applying the same techniques of *insertion of unities*[5] as in non-fermionic path integrals and subsequently taking the limit $N\to\infty$, one obtains, after integration of the momenta, the formal result[11]

$$\boxed{\Gamma^{11}[\mathbf{A}] = \frac{i}{2}\int_0^{\infty}\frac{dT}{T}\oint_P D\mathbf{x}\oint_A D\boldsymbol{\psi}\; e^{-\int_0^T d\tau\;\mathcal{L}(\mathbf{x},\dot{\mathbf{x}},\boldsymbol{\psi},\dot{\boldsymbol{\psi}})} \quad ,} \tag{4.13}$$

whereas

$$\boxed{\mathcal{L}(\mathbf{x},\dot{\mathbf{x}},\boldsymbol{\psi},\dot{\boldsymbol{\psi}}) := \frac{\dot{\mathbf{x}}^2}{4} + \frac{1}{2}\boldsymbol{\psi}\dot{\boldsymbol{\psi}} - ie\dot{\mathbf{x}}\mathbf{A} + ie\psi^\mu\psi^\nu F_{\mu\nu} + m^2 \quad .} \tag{4.14}$$

The $\boldsymbol{\psi} = (\psi^1, .., \psi^4)$ are 4-dimensional, real-Grassmann valued paths. They actually result from the Grassmann variables $\vartheta^1, \vartheta^2, \overline{\vartheta}^1, \overline{\vartheta}^2$ by means of a coordinate transformation[4]. The integrals $\oint_P$ and $\oint_A$ denote periodic conditions $\mathbf{x}(0) = \mathbf{x}(T)$ for the real integral and antiperiodic conditions $\boldsymbol{\psi}(0) = -\boldsymbol{\psi}(T)$ for the Grassmann integral. This periodicity and antiperiodicity results from the symmetric and antisymmetric form of the diagonal elements in (4.11).

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[4] We denote $\boldsymbol{\vartheta} := (\vartheta^1, \vartheta^2)$ and $\overline{\boldsymbol{\vartheta}} := (\overline{\vartheta}1, \overline{\vartheta}^2)$.

[5] Note the representation of the unity

$$\mathbb{1} = i\int d^4\mathbf{x}\; d\vartheta^1\; d\vartheta^2\;\; |\mathbf{x},\boldsymbol{\vartheta}\rangle\langle\mathbf{x},\boldsymbol{\vartheta}| \quad . \tag{4.12}$$

4.2 N-point functions

With the effective action (4.13) at hand, we are now ready to look at the one-loop contributions to the 1PI-N-point functions, namely

$$\Gamma_N^{11,\ \mu_1..\mu_N}(\mathbf{x}_1,..,\mathbf{x}_N) := \frac{\delta\Gamma^{11}}{\delta A_{\mu_1}(\mathbf{x}_1)..\delta A_{\mu_N}(\mathbf{x}_N)}\Big|_{\mathbf{A}=0}$$

$$= \frac{i}{2}(ie)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint_P D\mathbf{x} \oint_A D\boldsymbol{\psi}\ e^{-\int_0^T d\tau\left[\frac{\dot{\mathbf{x}}^2}{4}+\frac{1}{2}\boldsymbol{\psi}\dot{\boldsymbol{\psi}}\right]}$$

$$\times \prod_{k=1}^N \left[\dot{x}^{\mu_k}(\mathbf{x}(\tau_k)) - 2\psi^\mu \psi^{\mu_k}\partial_\mu\right]\delta(\mathbf{x}(\tau_k)) - \mathbf{x}_k) \quad . \tag{4.15}$$

Performing a Fourier-transformation of (4.15) yields

$$\widetilde{\Gamma}_N^{11,\mu_1,..,\mu_N}(\mathbf{p}^1,..,\mathbf{p}^N) = \frac{i}{2}\frac{(ie)^N}{(2\pi)^{DN/2}} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint_P D\mathbf{x}\ e^{-\int_0^T \frac{\dot{\mathbf{x}}^2}{4} d\tau} \oint_A D\boldsymbol{\psi}\ e^{-\int_0^T \frac{d\tau}{2}\boldsymbol{\psi}\dot{\boldsymbol{\psi}}}$$

$$\times \prod_{k=1}^N \int_0^T d\tau_k \Big[\underbrace{\dot{x}^{\mu_k}(\tau_k)}_{=:x_k^{\mu_k}} + 2i(\underbrace{\psi^{\mu_k}(\tau_k)}_{=:\psi_k^{\mu_k}})(\mathbf{p}^k\boldsymbol{\psi}(\tau_k))\Big] \cdot e^{i\mathbf{p}^k \mathbf{x}(\tau_k)} \quad , \tag{4.16}$$

whereas we formally replaced $4 \to D$, preparing for dimensional regularization. Note that $\widetilde{\Gamma}_N^{11,\mu_1..,\mu_N}$ is a contravariant tensor field. Its behavior is thus characterized by how it acts on arbitrary covectors $\boldsymbol{\varepsilon}^1,..,\boldsymbol{\varepsilon}^N$ and it suffices to determine the contractions

$$\boldsymbol{\varepsilon}^1_{\mu_1}\dots\boldsymbol{\varepsilon}^N_{\mu_N}\widetilde{\Gamma}_N^{\mu_1..\mu_N}(\mathbf{p}^1,..,\mathbf{p}^n) \quad . \tag{4.17}$$

Now to further evaluate (4.16), or (4.17) for that matter, it would be useful to express the integrand purely as an exponential. We therefore introduce the Grassmann variables[6] $\vartheta_k, \overline{\vartheta}_k$ and write[11]

$$\boldsymbol{\varepsilon}^k\dot{\mathbf{x}}_k + 2i(\boldsymbol{\varepsilon}^k\boldsymbol{\psi}_k)(\mathbf{p}^k\boldsymbol{\psi}_k) = \int d\overline{\vartheta}_k d\vartheta_k\ \exp\left[\overline{\vartheta}_k\vartheta_k(\boldsymbol{\varepsilon}^k\dot{\mathbf{x}}_k) + \sqrt{2}\vartheta_k(\boldsymbol{\varepsilon}^k\boldsymbol{\psi}_k) + \sqrt{2}i\overline{\vartheta}_k(\mathbf{p}^k\boldsymbol{\psi}_k)\right] \quad . \tag{4.18}$$

Evaluating the resulting path integrals finally yields

$$\boldsymbol{\varepsilon}^1_{\mu_1}\dots\boldsymbol{\varepsilon}^N_{\mu_N}\widetilde{\Gamma}_N^{\mu_1..\mu_N}(\mathbf{p}^1,..,\mathbf{p}^n) = i\frac{2\pi^{\frac{D}{2}}(ie)^N}{(2\pi)^{DN/2}}\cdot\delta\left[\sum_{k=1}^N \mathbf{p}^k\right]\cdot\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}}e^{-m^2T}\prod_{j=1}^N\left[\int_0^T d\tau_j \int d\overline{\vartheta}_j d\vartheta_j\right]$$

$$\exp\left[\frac{1}{2}\sum_{k,l=1}^N\left[\mathbf{p}^k\mathbf{p}^l - 2i\overline{\vartheta}_l\vartheta_l\boldsymbol{\varepsilon}^l\mathbf{p}^k\partial_1 + \overline{\vartheta}_k\vartheta_k\overline{\vartheta}_l\vartheta_l\boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^l\partial_1^2\right]G_B(\tau_k,\tau_l)\right]$$

$$\times\exp\left[\frac{1}{2}\sum_{k,l=1}^N\left[\overline{\vartheta}_k\overline{\vartheta}_l\mathbf{p}^k\mathbf{p}^l - \vartheta_k\vartheta_l\boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^l - 2i\overline{\vartheta}_k\vartheta_l\mathbf{p}^k\boldsymbol{\varepsilon}^l\right]G_F(\tau_k,\tau_l)\right] \quad . \tag{4.19}$$

One should not be irritated by the fact that the $\boldsymbol{\varepsilon}^k$, as arguments of a multilinear form on the left hand side, appear in an exponent on the right hand side of (4.19). Due to the Grassmann numbers included, only terms of linear order eventually remain.

[6] Using the convention $\int d\overline{\vartheta}_k\ d\vartheta_k\ \overline{\vartheta}_k\vartheta_k = 1$.

The G_B & G_F are the Green's functions for the differential operators $2\frac{d^2}{d\tau^2}$ & $2\frac{d}{d\tau}$, acting on functions on the interval $[0,T]$, with periodic & antiperiodic boundary conditions respectively[7]. They have the form:

$$G_B(\tau_1,\tau_2) = |\tau_1-\tau_2| - \frac{(\tau_1-\tau_2)^2}{T} \quad ,$$

$$G_F(\tau_1,\tau_2) = \mathrm{sgn}(\tau_1-\tau_2) \quad . \tag{4.20}$$

It should be noted that the classical action term in (4.3), depends only linearly on the gauge field $\mathbf{A}$. Thus, the one-loop contribution (4.19) to any 1PI-N-point function, is for $N \geq 2$ exactly that 1PI-N-point function. Having now at hand an integral expression for the 1PI-N-point functions, we can proceed to looking more carefully at one important special case.

4.3 The vacuum polarization tensor

One important special case of N-point function, is the so-called *polarization tensor*, appearing as a second order expansion term in the S matrix of interacting QED. It is defined as[10]

$$\frac{i}{4\pi}\Pi^{\mu\nu}(\mathbf{x},\mathbf{y}) := (ie)^2 \cdot \hat{\bar{\psi}}(\mathbf{x})\gamma^\mu\hat{\psi}(\mathbf{x})\hat{\bar{\psi}}(\mathbf{y})\gamma^\nu\hat{\psi}(\mathbf{y}) \tag{4.21}$$

and corresponds to the interaction of a propagating photon with the *vacuum*, by means of electron/positron creation and annihilation. The *perturbed* photon propagator reads[17]

$$iD^{\mathrm{int}}_{F,\varkappa\lambda}(\mathbf{x},\mathbf{y}) := \langle 0|\, T\left[\hat{A}_\varkappa(\mathbf{x})S\hat{A}_\lambda(\mathbf{y})\right]|0\rangle$$

$$= iD_{F,\varkappa\lambda}(\mathbf{x},\mathbf{y}) + \left[iD_{F,\varkappa\mu} * \frac{i\Pi^{\mu\nu}}{4\pi} * iD_{F,\nu\lambda}\right](\mathbf{x},\mathbf{y}) + (\dots) \tag{4.22}$$

with

$$iD_{F,\varkappa\lambda}(\mathbf{x},\mathbf{y}) := \langle 0|\, T\left[\hat{A}_\varkappa(\mathbf{x})\hat{A}_\lambda(\mathbf{y})\right]|0\rangle \tag{4.23}$$

as *free* photon propagator. The vacuum polarization term corresponds to the Feynman diagram (4.1) and is of 1st order in the fine-structure constant $\alpha := \frac{e^2}{4\pi}$.

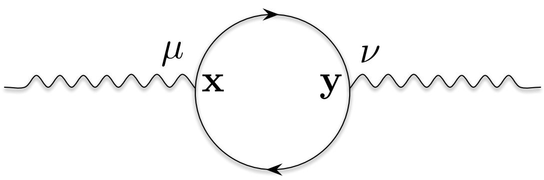

Figure 4.1: Vacuum polarization diagram in QED.

It can be shown that the polarization tensor is given by the 2-point function[4]

$$\widetilde{\Gamma}^{\mu\nu}_2(\mathbf{p}^1,\mathbf{p}^2) = (2\pi)^{\frac{D}{2}} \cdot \delta(\mathbf{p}^1+\mathbf{p}^2) \cdot \widetilde{\Pi}^{\mu\nu}(\mathbf{p}^1) \quad . \tag{4.24}$$

This special structure for the 1PI-2-point function $\widetilde{\Gamma}^{\mu\nu}_2$ is in total accordance with expression (4.19), actually resulting from the fact that $\Gamma^{\mu\nu}_2$ depends only on the difference of its arguments! The evaluation of the Grassmann integrals in (4.19) is straightforward and results in

$$\widetilde{\Pi}^{\mu\nu}(\mathbf{p}) = -\frac{i}{(2\pi)^{\frac{D}{2}}}\frac{8e^2}{(4\pi)^{\frac{D}{2}}} \cdot \left[\delta^{\mu\alpha}p_\alpha\delta^{\nu\beta}p_\beta - (\mathbf{p})^2\delta^{\mu\nu}\right] \cdot \Gamma\left(2-\tfrac{D}{2}\right) \cdot \int_0^1 du\ u(1-u)\left[m^2+(\mathbf{p})^2u(1-u)\right]^{\frac{D}{2}-2} \quad . \tag{4.25}$$

[7]Actually $2\frac{d^2}{d\tau^2}$ acting on the equivalence classes of paths, with respect to the equivalence relation $\mathbf{y} \sim \mathbf{z} \ :\Leftrightarrow\ (\mathbf{y}-\mathbf{z}) = \mathrm{const}.$

Note that this result is still in Euclidean form: Rotating back to Minkowskian, one obtains

$$\boxed{\widetilde{\Pi}_M^{\mu\nu}(\mathbf{p}) = \frac{1}{(2\pi)^{\frac{D}{2}}}\frac{8e^2}{(4\pi)^{\frac{D}{2}}}\cdot\left[p^\mu p^\nu - g(\mathbf{p},\mathbf{p})\cdot g^{\mu\nu}\right]\cdot\Gamma\left(2-\tfrac{D}{2}\right)\cdot\int_0^1 du\; u(1-u)\left[m^2 - g(\mathbf{p},\mathbf{p})u(1-u)\right]^{\frac{D}{2}-2}\quad.} \tag{4.26}$$

The same result one obtains, when directly calculating the vacuum polarization contribution to the QED S matrix, using Wicks theorem in its Dyson expansion[17]. Observe that this expression is divergent for $D \to 4$ and needs to be handled by means of renormalization.

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