Physics of Planetary Systems — Exercises

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Solutions to Set 6

Problem 6.1

Advantages:

- No bias for nearby stars or planets around solar-type stars.
- Sensitive to Earth-mass planets using ground-based observations: one of few methods that can do this.
- Most sensitive for planets in the "lensing zone", $0.6 < a < 2$ AU for stars in the bulge. This is the habitable zone.
- Multiple systems can be detected at the same time.
- Detection of free-floating planets possible.
- Can get good statistics on Earth mass planets in the habitable zone of stars.

Disadvantages:

- Probability of lensing events small.
- One time event, no possibility to confirm, or improve measurements.
- Duration of events is hours to days. Need coordinated observations from many observatories.
- Planets are distant: so, no detailed studies of the planet are possible.
- Planet hosting star is distant: detailed studies of the host star very dfficult.
- Precise orbital parameters of the planet not possible.
- Light curves are complex: only one crossing of the caustic. No unique solution and often a non-planet can also model the light curves.

Problem 6.2

We first need to compute the Einstein Radius, $\theta_{\rm E}$,

$$\theta_{\rm E} = \sqrt{\frac{4GM}{c^2}\frac{D_{\rm LS}}{D_{\rm L}D_{\rm S}}}$$

We then need to calculate the magnification from:

$$\mu = \frac{u^2+2}{u\sqrt{u^2+4}},$$

where u is defined as $u \equiv \beta/\theta_{\rm E}$, and β is the impact parameter in radians. The duration of the event is given by

$$t = \frac{R_{\rm E}}{v},$$

with $R_{\rm E}$ being the the projected Einstein Radius. The involved distances are $D_{\rm L} = 2$ kpc, $D_{\rm S} = 10$ kpc, $D_{\rm LS} = 8$ kpc.

a) Assuming $M = 1\ M_\odot$, we obtain $\theta_{\rm E} = 8.77 \times 10^{-9}$ rad $= 0.00181'' = 1.81$ mas, $u = 0.01/1.81 = 0.00552$, and thus, $\underline{\mu = 181}$. With $R_{\rm E} = \theta_{\rm E} D_{\rm L} = 5.4 \times 10^{13}$ cm and an assumed velocity $v \approx 200$ km/s, the transit duration is $\underline{t = 31.2\ \rm d}$.

b) $M = 1\ M_{\rm Jup}$ leads to: $\theta_{\rm E} = 2.7 \times 10^{-10}$ rad $= 0.56$ mas, $u = 0.01/0.56 = 0.18$, $\underline{\mu = 5.63}$, $R_{\rm E} = 1.66 \times 10^{12}$ cm, $\underline{t = 23.1\ \rm h}$.

c) $M = 1\ M_\oplus$ leads to: $\theta_{\rm E} = 1.5 \times 10^{-11}$ rad $= 0.00314$ mas, $u = 0.01/0.00314 = 3.18$, $\underline{\mu = 1.01}$, $R_{\rm E} = 9.24 \times 10^{10}$ cm, $\underline{t = 4.6\ \rm h}$.

Problem 6.3

The gas drag force (as usual, in the Epstein regime) is given by

$$F_{\rm gas} = \frac{4}{3} \rho c_{\rm s} \sigma v$$

where $\sigma = \pi s^2$, and v is the relative velocity of the planetesimal with respect to gas:

$$v = \eta v_{\rm K}, \qquad \eta \approx \frac{c_{\rm s}^2}{v_{\rm K}^2}$$

The velocities $c_{\rm s}$ and $v_{\rm K}$ were already calculated in a previous problem:
$c_{\rm s} \approx 2 \rm kms^{-1}$ and $v_{\rm K} \approx 30 \rm kms^{-1}$, so that $\eta \approx (2/30)^2 \approx 1/200$.
Substituting other numerical values (in CGS!) results in

$$F_{\rm gas} \approx \frac{4}{3} \cdot 10^{-9} \cdot 2 \cdot 10^5 \cdot (3s^2) \cdot 30 \cdot 10^5 \cdot \left(\frac{1}{200}\right) \approx 8 \cdot 10^{-4} s^2 \cdot 1.5 \cdot 10^4 \approx 10 s^2.$$

The mutual gravitational force acting upon two planetesimals with radius s is strongest "at contact" and then given by

$$F_{\rm grav} = \frac{Gm^2}{(2s)^2} = \frac{G}{(2s)^2} \left(\frac{4}{3}\pi \rho_{\rm plan} s^3\right)^2 \approx \frac{G}{4s^2} \left(4\rho_{\rm plan} s^3\right)^2 \approx 4G\rho_{\rm plan}^2 s^4$$

or, numerically, assuming $\rho_{\rm plan} \approx 2$,

$$F_{\rm grav} \approx 4 \cdot 7 \cdot 10^{-8} \cdot 4 s^4 \approx 10^{-6} s^4.$$

Equating $F_{\rm gas}$ and $F_{\rm grav}$ leads to

$$10 s^2 = 10^{-6} s^4$$

or

$$s = 3 \cdot 10^3 \text{ cm} = 30 \text{ m}.$$

Therefore, gravity seems to be important already at sizes $\ll 1$ km, but: gas drag acts permanently, whereas mutual gravity only during (short-lasting) close encounters.
What is more, the case where two like-sized bodies stick together is very rare. Typically smaller objects stick to a bigger one, in which case the acceleration due to gas drag is stronger. We can, for example, calculate how big the big object had to be, in order to bind a small object with a radius of one meter to its surface:

$$F_{\rm grav} = \frac{G m_{\rm big} m_{\rm small}}{s_{\rm big}^2} = \frac{4}{3} \rho_{\rm gas} c_{\rm s} \sigma_{\rm small} v = F_{\rm gas}.$$

Again, with $\sigma_{\text{small}} = \pi s^2_{\text{small}}$ and $m = \frac{4}{3}\pi\rho_{\text{plan}} s^3$, we find

$$s_{\text{big}} = \frac{3c_s v \rho_{\text{gas}}}{4\pi\rho^2_{\text{plan}} G s_{\text{small}}} \approx 260 \text{ m.}$$

This value is already higher by an order of magnitude.

Problem 6.4

Rebounds are possible if the typical velocity of fragments v is less than the escape velocity of the debris cloud emerged after the collision of two planetesimals of radius s. Roughly,

$$v < v_{\text{esc}} \sim \sqrt{\frac{2Gm}{s}} \sim \sqrt{\frac{2G(4/3)\pi\rho_{plan}s^3}{s}} \sim \sqrt{8G\rho_{plan}}s$$

whence

$$s > \frac{v}{\sqrt{8G\rho_{plan}}} \sim \frac{10^3}{\sqrt{8\cdot 7\cdot 10^{-8}\cdot 2}} \sim \frac{10^3}{\sqrt{10^{-6}}} \sim 10^6\text{cm} \sim 10\text{km}.$$

Problem 6.5

Let a grain move through an ensemble of background grains with a volume number density n_0. Assuming a collision cross section $\sigma = \pi(s+s_0)^2$ with object radii s and s_0, the chance that the grain hits a background grain while it moves a certain distance dx is given by

$$\mathrm{d}P = n_0\sigma \mathrm{d}x,$$

where the expression $n_0\sigma$ can be considered a "cross section density". (The integral version for the probability that it can travel a distance x is $P(x) = \mathrm{e}^{-xn_0\sigma}$.) Thus the mean free path λ_0 is defined through

$$n_0\sigma\lambda_0 \overset{!}{=} 1.$$

And with $s = s_0 = 1$ µm and $n_0 = 10^9$ m^{-3}, we find

$$\lambda_0 \approx 80\ m.$$

On the other hand, the *rate of collisions*, R, depends on the velocity at which the object moves through the others:

$$\mathsf{R} = n\sigma v_{\text{rel}} = \frac{v_{\text{rel}}}{\lambda}.$$

($v = \mathrm{d}x/st$). Using $v_{\text{rel}} = 0.001$ m s^{-1} and $\lambda = \lambda_0 = 80$ m, this rate is $\mathsf{R}_0 \approx 1$ per day (for collisions between 1-µm-sized grains).
In order to consider growth, we need to put more effort in the expression for σ and assume that $s > s_0 = 1$ µm. The accretion of one background grain onto our growing grain leads to a straight-forward increase of its mass by that of the background grain:

$$m' = m + m_0$$

and

$$\mathrm{d}m = m_0\mathsf{R}\mathrm{d}t = m_0 n_0\sigma v_{\text{rel}}\mathrm{d}t = m_0 n_0 \pi(s+s_0)^2 v_{\text{rel}}\mathrm{d}t.$$

In contrast, the behavior of an objects *size* depends on its dimensionality: $m \propto s^D$ or

$$m = m_0\left(\frac{s}{s_0}\right)^D.$$

Therefore,

$$\mathrm{d}m = D\frac{m_0}{s_0}\left(\frac{s}{s_0}\right)^{D-1}\mathrm{d}s$$

which leads to

$$\frac{D}{s_0}\left(\frac{s}{s_0}\right)^{D-1}\mathrm{d}s = n_0\pi(s+s_0)^2 v_{\mathrm{rel}}\mathrm{d}t.$$

Assuming that, most of the time, $s \gg s_0$, we can simplify the equation to

$$\frac{D}{s_0^3}\left(\frac{s}{s_0}\right)^{D-3}\mathrm{d}s = n_0\pi v_{\mathrm{rel}}\mathrm{d}t$$

and find (when integrating s from s_0 to s_{max})

$$\Delta t|_{D=3} = \frac{D}{D-2}\frac{1}{\pi s_0^2 n_0 v_{\mathrm{rel}}}\left[\left(\frac{s_{\mathrm{max}}}{s_0}\right)^{D-2} - 1\right] = \frac{D}{D-2}\frac{4}{\mathsf{R}_0}\left[\left(\frac{s_{\mathrm{max}}}{s_0}\right)^{D-2} - 1\right]$$

With $s_{\mathrm{max}}/s_0 = 1000$, $\mathsf{R}_0^{-1} \approx 1$ day, and $D = 3$, the result is

$$\Delta t|_{D=3} \approx \frac{12}{\mathsf{R}_0}\frac{s_{\mathrm{max}}}{s_0} \approx 12000 \text{ days} \approx 30 \text{ yr}.$$

For $D = 2$, we can either convince ourselves that

$$\frac{1}{D-2}\left[\left(\frac{s_{\mathrm{max}}}{s_0}\right)^{D-2} - 1\right] \xrightarrow{D\to 2} \ln\left(\frac{s_{\mathrm{max}}}{s_0}\right)$$

or re-do the integral over s. In either case, one obtains

$$\Delta t|_{D=2} = \frac{8}{\mathsf{R}_0}\ln\left(\frac{s_{\mathrm{max}}}{s_0}\right)$$

and

$$\Delta t|_{D=2} \approx \ln 1000 \cdot 8 \text{ days} \approx 60 \text{ days}.$$

As long as the relative velocities are low enough for the grains to stay porous (and to not get compactified), the growth is rather rapid.