Physics of Planetary Systems — Exercises

Astrophysikalisches Institut und Universitätssternwarte Jena
Thüringer Landessternwarte Tautenburg

2009-05-14

Solutions to Set 3

Problem 3.1

Here comes a list of four possible sources of false positives.

- Grazing eclipse by a binary: can easily be distinguished with radial velocity measurements which would show an amplitude of several tens of km/s instead of hundreds of m/s.
- Transit of a main sequence star across a giant. A spectra of the host star should reveal it is a giant. Plus the transit duration will be too long. A transit across a giant star can take many tens of hours to days.
- Eclipsing binary in background diluted by the light of a bright foreground object. This is difficult to resolve with radial velocity measurements. Probably need very high resolution imaging, or spectra in the infrared.
- Hierarchical binary, i.e. an eclipsing binary in orbit around a brighter star. High resolution imaging is needed to resolve system, or infrared measurements. Depending on the orbital period of the binary about the main star, one could see a radial velocity trend due to a binary star.

Problem 3.2

The condition to be met is

$$\left(\frac{R_{\rm planet}}{R_{\rm star}}\right)^2 = 1\%,$$

from which we find

$$R_{\rm star} = \frac{R_{\rm planet}}{\sqrt{1\%}} = 10 R_{\rm planet}.$$

Since we mistakenly asked for 1% photometric amplitude (instead of 0.1% as planned) and $R_{\rm planet} = 1\ R_{\rm Jup}$, we obtain

$$R_{\rm star} = 10\ R_{\rm Jup} = 1\ R_{\rm sun}.$$

So, we end up with a star of solar radius and, thus, a G2 star.

Problem 3.3

(a) The transit probability is just $p = R_{\rm star}/a$. With $a = 0.1$ AU $= 21.4\ R_{\rm sun}$ and $R_{\rm star} = R_{\rm sun}$, we obtain $p = 0.046$. Neptune has a radius that is 0.07 times that of the sun. The photometric amplitude is thus $0.07^2 = 0.005 = 0.5\%$. In class, an expression for the transit duration was given:

$$\tau = 2R_{\rm star}\left[\frac{P}{2\pi G\mathscr{M}_{\rm star}}\right]^{1/3} = 1.82 \text{ hours} \times \frac{R_{\rm star}}{R_{\rm sun}}\left[\frac{P}{1 \text{ day}}\frac{\mathscr{M}_{\rm sun}}{\mathscr{M}_{\rm star}}\right]^{1/3},$$

which can also be expressed as

$$\tau = 2R_{\rm star}\left[\frac{a}{G\mathscr{M}_{\rm star}}\right]^{1/2} = 13 \text{ hours} \times \frac{R_{\rm star}}{R_{\rm sun}}\left[\frac{a}{1 \text{ AU}}\frac{\mathscr{M}_{\rm sun}}{\mathscr{M}_{\rm star}}\right]^{1/2}.$$

These expressions can be easily derived using Keplers laws and assuming circular orbits.

For calculating the transit duration, we first assume an orbital inclination $i \approx 0$ (i.e, you are looking in the plane of the orbit) and $R_{\rm star} \gg R_{\rm planet}$. From $a = 0.1$ AU we find $P = 0.032$ years $= 11.6$ days. Thus $\tau = 4.1$ hours.

(b) For a start, we need to know the radius of a K0III star, which can range from 8 to 20 solar radii. We will use an intermediate value, $R_{\rm star} = 15\ R_{\rm sun}$. Thus, the transit probability for our case is

$$p = R_{\rm star}/a = \frac{15 \cdot 7 \times 10^{10} \text{ cm}}{3 \times 10^{13} \text{ cm}} = 0.035.$$

The photometric amplitude is given by

$$\frac{\Delta I}{I} = (1/15)^2 = 0.004,$$

i.e. this looks like a transiting Neptune! In order to calculate the transit duration, we need to assume a stellar mass. Masses of giant stars are now well known and can span 1–2 $\mathscr{M}_{\rm sun}$. Let us assume a solar mass for the moment. Hence, $a = 2$ AU implies an orbital period $P = 2.82$ years $= 1030$ days. The transit duration is thus $\tau = 276$ hours $= 11.5$ days, i.e. we now know it is not a transiting Neptune! Even if we assumed $R = 1\ R_{\rm sun}$, we still get a transit time of 18.4 hrs. Thus the transit duration can be used to get an estimate of how big your star is. (For the mass (1.7 $\mathscr{M}_{\rm sun}$) and radius (9 $R_{\rm sun}$) of the K0IIIb star Pollux, we obtain $p = 0.021$, $\Delta I/I = 0.012$, and $\tau = 127$ hours $= 5.3$ days.)

Problem 3.4

Assuming, say, a temperature of 1000 K at 1 AU from the Sun gives the sound velocity

$$c_{\rm s} = \sqrt{\frac{kT}{\mu m_{\rm p}}} \sim \sqrt{\frac{1.4 \cdot 10^{-16} \cdot 1000}{2 \cdot 1.7 \cdot 10^{-24}}} \sim \sqrt{5 \cdot 10^{10}} \sim 2 \cdot 10^5 \sim 2 \text{ km s}^{-1}.$$

The Kepler circular velocity at the same distance from the Sun is given by

$$v_{\rm K} = \sqrt{G\mathscr{M}_\star/r} \approx 30 \text{ km s}^{-1}.$$

Obviously, at 10 or 100 AU the inequality $c_{\rm s} \ll v_{\rm K}$ holds as well.

Problem 3.5

Assume power laws

$$c_{\rm s}^2 \propto T \propto r^{-\xi} \qquad \text{and} \qquad \Sigma \propto r^{-\eta},$$

so that

$$\nu = \alpha \frac{c_{\rm s}^2}{\Omega_{\rm K}} \propto r^{-\xi+3/2}.$$

Substitute these into the formula for the radial velocity

$$v_r = -\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}\left(\Sigma \nu \sqrt{r}\right)$$

to get

$$v_r \propto r^{\eta-1/2} \cdot r^{-\eta-\xi+3/2+1/2-1} \propto r^{-\xi+1/2}$$

Now, the stationary continuity equation

$$\frac{\partial(\Sigma r v_r)}{\partial r} = 0$$

requires $\Sigma r v_r = \text{const}$, or

$$r^{-\eta} \cdot r \cdot r^{-\xi+1/2} = \text{const}$$

whence

$$\eta = -\xi + 3/2.$$

Therefore, a general solution is

$$T \propto r^{-\xi}, \qquad \nu \propto r^{-\xi+3/2}, \qquad \Sigma \propto r^{\xi-3/2}.$$

To be “physical”, these solutions must have at least $\xi > 0$ (the farther out from the star, the colder). On the other hand, $\xi < 3/2$ is a reasonable requirement (the surface density is not expected to grow outward). These are not strict limitations though.

Plotting several of these solutions, for instance for $\xi = 0$, $1/2$, 1, and $3/2$ is straightforward. Hopefully you will not do that in linear scale... log-log is the most natural scale to plot power laws.

Addendum: To better understand the concept of viscosity, it can be insightful to study the outward transport of angular momentum trough a viscous disk.

The dynamical viscosity η is defined through the frictional force per area (= stress tension) between two layers spaced by a distance Δx with relative parallel velocity Δv:

$$\frac{F}{A} = \eta \frac{\Delta v}{\Delta x},$$

and straight-forward differential version reads

$$\frac{F}{A} = \eta \frac{\mathrm{d}v}{\mathrm{d}x},$$

where the Kepler velocity is given by,

$$v = \sqrt{\frac{G\mathcal{M}}{r}}$$

the viscosity can be expressed as,

$$\eta = \rho \nu = \rho \alpha c_\mathrm{s} h,$$

and the contact area of the viscous layers reads

$$A = 2\pi r h.$$

In addition the disk’s scale height h is given by

$$h = \frac{c_\mathrm{s}}{\Omega_\mathrm{k}}.$$

Here Ω_k is the Keplerian angular frequency,

$$\Omega_\mathrm{k} = \sqrt{\frac{G\mathcal{M}_\odot}{r^3}},$$

and c_s the sound speed,

$$c_\mathrm{s} = \sqrt{\frac{kT}{\mu m_\mathrm{p}}}.$$

What remains to be known is the local mass density ρ, which is related to the surface mass density Σ through

$$\rho(r) = \frac{\Sigma(r)}{h(r)}.$$

Since we know $\Sigma \propto r^{-1}$, we can write $\Sigma = \Sigma_0 r_0/r$ and

$$\mathscr{M}_{\text{disk}} \equiv \int\limits_0^{R_{\text{disk}}} \Sigma(r) 2\pi r \mathrm{d}r = 2\pi\Sigma_0 r_0 \int\limits_0^{R_{\text{disk}}} \mathrm{d}r = 2\pi\Sigma_0 r_0 R_{\text{disk}},$$

whence we find

$$\rho(r) = \frac{\mathscr{M}_{\text{disk}}}{2\pi r h R_{\text{disk}}}.$$

Finally, the force is related to the angular momentum transfer via

$$\dot{L} = rF = r\eta A \frac{\mathrm{d}v}{\mathrm{d}x}$$

and we find

$$\begin{aligned}
\dot{L} &= r(\rho\alpha c_{\text{s}} h)(2\pi r h)\left(-\frac{1}{2}\sqrt{\frac{G\mathscr{M}}{r^3}}\right) \\
&= -\pi\alpha\rho c_{\text{s}} h^2 \sqrt{G\mathscr{M} r} \\
&= -\pi\alpha \left(\frac{\mathscr{M}_{\text{disk}}}{2\pi r h R_{\text{disk}}}\right) c_{\text{s}} h^2 \sqrt{G\mathscr{M} r} \\
&= -\pi\alpha \left(\frac{\mathscr{M}_{\text{disk}}}{2\pi r R_{\text{disk}}}\right) \frac{c_{\text{s}}^2}{\Omega_{\text{k}}} \sqrt{G\mathscr{M} r} \\
&= -\pi\alpha \left(\frac{\mathscr{M}_{\text{disk}}}{2\pi r R_{\text{disk}}}\right) \left(\frac{kT}{\mu m_{\text{p}}}\right) r^2 \\
&= -\pi\alpha \left(\frac{r\mathscr{M}_{\text{disk}}}{2\pi R_{\text{disk}}}\right) \left(\frac{kT}{\mu m_{\text{p}}}\right).
\end{aligned}$$

Now, we can use standard estimates for the α parameter ($\sim 10^{-3}$), the temperature ($T \sim 100$ K) and the molecular weight ($\mu \approx 2$). If we consider $r = R_{\text{disk}}$ for convenience, we find

$$\dot{L} \approx 4 \times 10^{31} \text{ Nm}.$$

For the Sun, we have

$$L_\odot \sim \frac{2}{5} \frac{2\pi \mathscr{M}_\odot R_\odot^2}{P_\odot} \approx 1 \times 10^{42} \text{ Nms}.$$

(with $P \approx 25\ d = 25 \cdot 86400$ s). So, we can write

$$\dot{L} \approx 4 \times 10^{-11} L_\odot \text{ s}^{-1} \sim \underline{\underline{10^{-3}\ L_\odot \text{ yr}^{-1}}}.$$

Thus, the mechanism can transport a significant amount of angular momentum outward within a few thousand or tens of thousands of years.