General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 12 - Solutions

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Problem 01

Starting with

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad , \quad g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + \mathcal{O}(h^2)$$

whereas

$$h^{\alpha\beta} := \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}$$

it follows

$$\Gamma^{\nu}_{\varkappa\lambda} = \frac{g^{\nu\rho}}{2}\left(\partial_\varkappa g_{\rho\lambda} + \partial_\lambda g_{\varkappa\rho} - \partial_\rho g_{\varkappa\lambda}\right) = \frac{\eta^{\nu\rho}}{2}\left(\partial_\varkappa h_{\rho\lambda} + \partial_\lambda h_{\varkappa\rho} - \partial_\rho h_{\varkappa\lambda}\right) + \mathcal{O}(h^2) \tag{0.1}$$

From

$$\Box x^\nu = g^{\varkappa\lambda}\nabla_\varkappa\nabla_\lambda x^\nu = g^{\varkappa\lambda}\left(\partial_\varkappa - \Gamma^{\rho}_{\varkappa\lambda}\right)\underbrace{\nabla_\rho x^\nu}_{\partial_\rho x^\nu = \delta^\nu_\rho} = -g^{\varkappa\lambda}\Gamma^{\nu}_{\varkappa\lambda}$$

$$\overset{(0.1)}{=} -\frac{1}{2}\eta^{\varkappa\lambda}\eta^{\nu\rho}\left(\partial_\varkappa h_{\rho\lambda} + \partial_\lambda h_{\varkappa\rho} - \partial_\rho h_{\varkappa\lambda}\right) + \mathcal{O}(h^2) = -\eta^{\varkappa\lambda}\eta^{\nu\rho}\partial_\varkappa h_{\lambda\rho} + \frac{1}{2}\eta^{\varkappa\lambda}\eta^{\nu\rho}\partial_\rho h_{\varkappa\lambda} + \mathcal{O}(h^2)$$

$$= -\partial_\varkappa\underbrace{\eta^{\varkappa\lambda}\eta^{\nu\rho}h_{\lambda\rho}}_{=:h^{\varkappa\nu}} + \frac{\eta^{\nu\rho}}{2}\partial_\rho\underbrace{\eta^{\varkappa\lambda}h_{\varkappa\lambda}}_{=:h} + \mathcal{O}(h^2)$$

one can see the equivalence

$$\Box x^\nu = 0 \quad \Leftrightarrow \quad g^{\varkappa\lambda}\Gamma^{\nu}_{\varkappa\lambda} = 0 \quad \Leftrightarrow \quad \partial_\varkappa h^{\varkappa\nu} - \frac{\eta^{\varkappa\nu}}{2}\partial_\varkappa h = 0 + \mathcal{O}(h^2) \tag{0.2}$$

With the definition

$$\overline{h}^{\varkappa\nu} := h^{\varkappa\nu} - \frac{\eta^{\varkappa\nu}}{2}h$$

it is clear that

$$\boxed{\Box x^\nu = 0 \quad \Leftrightarrow \quad \partial_\varkappa \overline{h}^{\varkappa\nu} = 0 + \mathcal{O}(h^2)} \tag{0.3}$$

Problem 02

(a) Due to symmetry, the two objects are always located at symmetrical positions $\pm x(t)$ about the origin, while according to Newton

$$\ddot{x} = -\frac{MG}{(2x)^2}$$

Substituting $v := \dot{x}$ one obtains the start-value problem

$$v'v = \frac{dv}{dx}\frac{dx}{dt} = \dot{v} = \ddot{x} = -\frac{MG}{4x^2} \quad , \quad v(\infty) = 0$$

which solves as

$$v(x) = -\sqrt{\frac{MG}{2x}}$$

Finally, the start-value-problem

$$\dot{x}(t) = -\sqrt{\frac{MG}{2x}} \quad , \quad x(0) = 0$$

solves as

$$\int_0^x \sqrt{x}\, dx = -\sqrt{\frac{MG}{2}} \int_0^t dt \quad , \quad t \leq 0$$

that is

$$\boxed{x(t) = \left(\frac{9MG}{8}t^2\right)^{\frac{1}{3}}} \tag{0.4}$$

(b) The Newtonian approximation is only reasonable if the resulting equation of motion (EOM) of the objects reflects the one obtained from a general relativistic point of view, in particular the geodesic equation

$$\ddot{x}^1 = -\Gamma^1_{\varkappa\lambda}\dot{x}^\varkappa\dot{x}^\lambda \overset{!}{\approx} -\partial_x\Phi$$

where $\Phi = -MG/(2x)$ is the classical potential for one object in the others field. For low speeds, that is $\dot{x}^i \ll \dot{x}^0$, this is equivalent to

$$\partial_x\Phi \overset{!}{\approx} \underbrace{\Gamma^1_{00}}_{\in\mathcal{O}(h)} \cdot \underbrace{\dot{x}^0\dot{x}^0}_{\approx\frac{1}{|g_{tt}|}} \approx \Gamma^1_{00} \overset{(0.1)}{\approx} \partial_0 h_{01} - \frac{1}{2}\partial_1 h_{00}$$

For slowly changing fields, that is, $|\partial_0 h_{01}| \ll |\partial_1 h_{00}|$, one obtains

$$\boxed{h_{00} \approx -2\Phi} \tag{0.5}$$

For small $|h_{\mu\nu}| \ll 1$ the Newtonian EOM are up to zeroth order in h the same as if obtained from the Einstein field equations and geodetical EOM. But as the waves created are of order $\mathcal{O}(h)$ this is adequate, as any corrections would be of higher order in h!

(c) **The wave-equation**

Using (0.1) we obtain the Riemann-curvature tensor to first order in h:

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \mathcal{O}(h^2) \overset{(0.1)}{=} \frac{\eta^{\alpha\rho}}{2}\left[\partial_{\mu\beta}h_{\rho\nu} - \partial_{\nu\beta}h_{\mu\rho} - \partial_{\mu\rho}h_{\nu\beta} + \partial_{\nu\rho}h_{\mu\beta}\right] + \mathcal{O}(h^2) \tag{0.6}$$

and consequently the Ricci-tensor

$$R_{\beta\nu} = R^\alpha{}_{\beta\alpha\nu} \overset{(0.6)}{=} \frac{1}{2}\left[\partial_{\alpha\beta}h^\alpha{}_\nu + \partial_{\alpha\nu}h^\alpha{}_\beta - \partial_{\nu\beta}h - \Box h_{\beta\nu}\right] + \mathcal{O}(h^2) \tag{0.7}$$

The Ricci-scalar

$$R = \eta^{\beta\nu}R_{\beta\nu} + \mathcal{O}(h^2) \overset{(0.7)}{=} \partial_{\alpha\beta}h^{\alpha\beta} - \Box h + \mathcal{O}(h^2) \tag{0.8}$$

finally leads to the Einstein-tensor

$$G_{\beta\nu} = R_{\beta\nu} - \frac{\eta_{\beta\nu}}{2} R + \mathcal{O}(h^2) \overset{(0.7)}{\underset{(0.8)}{=}} \frac{\partial_\beta}{2}\left(\partial_\alpha h^\alpha{}_\nu - \partial_\nu \frac{h}{2}\right) + \frac{\partial_\nu}{2}\left(\partial_\alpha h^\alpha{}_\beta - \partial_\beta \frac{h}{2}\right)$$

$$- \frac{\Box}{2} \underbrace{\left(h_{\beta\nu} - \eta_{\beta\nu}\frac{h}{2}\right)}_{\overline{h}_{\beta\nu}} - \frac{1}{2}\eta^{\lambda\rho}\eta_{\beta\nu}\partial_\lambda \left(\partial_\alpha h^\alpha{}_\rho - \partial_\rho \frac{h}{2}\right) + \mathcal{O}(h^2) \qquad (0.9)$$

Choosing harmonic gauge condition $\Box x^\mu = 0$, that is

$$\eta^{\rho\nu}\left(\partial_\alpha h^\alpha{}_\rho - \partial_\rho \frac{h}{2}\right) \overset{(0.2)}{=} 0 + \mathcal{O}(h^2)$$

or equivalently[1]

$$\partial_\alpha h^\alpha{}_\rho - \partial_\rho \frac{h}{2} = 0 + \mathcal{O}(h^2) \qquad (0.10)$$

leads to

$$G_{\beta\nu} = -\frac{\Box}{2}\overline{h}_{\beta\nu} + \mathcal{O}(h^2) \qquad (0.11)$$

The Einstein-Field-Equations thus imply the wave-equation

$$\Box\overline{h}_{\mu\nu} = -16\pi G T_{\mu\nu} + \mathcal{O}(h^2) \qquad (0.12)$$

we shall be considered only to first order in h.

Solving the wave-equation for a single object

Any solution can be written in the form

$$\overline{h}_{\mu\nu}(x) = \int \overline{h}_{\mu\nu}(k) e^{ix^\lambda k_\lambda}\, dk \qquad (0.13)$$

while the vacuum equations $\Box\overline{h}_{\mu\nu} = 0$ imply

$$0 = \overline{h}_{\mu\nu}(k) \cdot \Box e^{ix^\lambda k_\lambda} = -\overline{h}_{\mu\nu}(k) \cdot k_\lambda k^\lambda$$

that is

$$\eta(\varkappa, \varkappa) = 0 \quad \forall\, \overline{h}_{\mu\nu}(k) \neq 0 \qquad (0.14)$$

Moreover, the harmonic gauge we demanded is equivalent to

$$0 = \overline{h}_{\mu\nu}(k) \cdot \partial_\mu e^{ix^\lambda k_\lambda} = ik_\mu \overline{h}^{\mu\nu}(k) \qquad (0.15)$$

that is, the spectral perturbation components $\overline{h}_{\mu\nu}(k)$ are perpendicular to the wave-vector k! .
As is known, the (retarded) Greens function of the D'Alembert-Operator $\Box$ is given by

$$G(x,y) = G(x-y) = -\frac{\delta\left(\|\mathbf{x}-\mathbf{y}\| - (x^0 - y^0)\right)}{4\pi\|\mathbf{x}-\mathbf{y}\|} \qquad (0.16)$$

and thus

$$\overline{h}^{ij}(x) \overset{(0.12)}{=} -16\pi G \int G(x,y) \cdot T^{ij}(y)\, dy \overset{(0.16)}{=} 4G \int \frac{T^{ij}\left(t - \|\mathbf{x}-\mathbf{y}\|, \mathbf{y}\right)}{\|\mathbf{x}-\mathbf{y}\|}\, d^3\mathbf{y}$$

[1] Note that $(\eta^{\rho\nu})$ is invertible.

We shall first consider a source, at each time-point t concentrated in a relatively narrow space-region $\mathbf{z}(t)$ (later $T \sim \delta$). **Assuming** that the source (or T for that matter) is moving at a speed much lower than 1, that is, the support $\operatorname{supp} f$ of

$$f(\mathbf{y}) := T(t - \|\mathbf{x} - \mathbf{y}\|, \mathbf{y})$$

is small with respect to $\|\mathbf{x}\|$, concentrated around $\mathbf{z}(t)$, implies

$$\overline{h}^{ij}(t, \mathbf{x}) \approx \frac{4G}{\|\mathbf{x} - \mathbf{z}(t)\|} \int T^{ij}\big(\underbrace{t - \|\mathbf{x} - \mathbf{z}(t)\|}_{\tau=\tau(t)}, \mathbf{y}\big)\ d^3\mathbf{y}$$

Partial integration leads to

$$\overline{h}^{ij}(t, \mathbf{x}) \approx \frac{4G}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \Big\{ \underbrace{\int \partial_{y^k}\left(y^i T^{kj}(\tau, \mathbf{y})\right)\ d^3\mathbf{y}}_{\substack{\text{surface integral}\\ =0}} - \int y^i \underbrace{\partial_k T^{kj}}_{-\partial_0 T^{0j}}(\tau, \mathbf{y})\ d^3\mathbf{y} \Big\} = \frac{4G}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \int y^i \partial_0 T^{0j}(\tau, \mathbf{y})\ d^3\mathbf{y}$$

$$\frac{4G}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \Big\{ \underbrace{\int \partial_{y^k}\left(y^j y^i \partial_0 T^{0k}(\tau, \mathbf{y})\right)\ d^3\mathbf{y}}_{\substack{\text{surface integral}\\ =0}} - \int y^j y^i \partial_0 \underbrace{\partial_k T^{0k}}_{-\partial_0 T^{00}}(\tau, \mathbf{y})\ d^3\mathbf{y} - \underbrace{\int y^j \partial_0 T^{0i}(\tau, \mathbf{y})\ d^3\mathbf{y}}_{\substack{\int y^i \partial_0 T^{0j}(\tau, \mathbf{y})\ d^3\mathbf{y} \\ \text{since } h^{ij} = h^{ji}}} \Big\}$$

$$= \frac{2G}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \int y^i y^j \partial_{00} T^{00}(\tau, \mathbf{y})\ d^3\mathbf{y} = \frac{2G}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \partial_{00} \underbrace{\int y^i y^j T^{00}\big|_{(\tau, \mathbf{y})}\ d^3\mathbf{y}}_{\sim\text{quadrupole moment}}$$

Similarly, by only performing one partial integration, we obtain

$$\overline{h}^{i0}(t, \mathbf{x}) \approx \frac{4G}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \int y^i \partial_0 T^{00}(\tau, \mathbf{y})\ d^3\mathbf{y} \tag{0.17}$$

Specifically, for an object moving slowly along a world-line $\mathbf{z}(t)$, that is, $T^{00}(t, \mathbf{y}) \approx M\delta^{(3)}(\mathbf{y} - \mathbf{z}(t))$, one gets

$$\overline{h}^{ij}(t, \mathbf{x}) \approx \frac{2GM}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \partial_{00}\left(z^i z^j\right)\big|_{\tau(t)} \tag{0.18}$$

and

$$\overline{h}^{i0}(t, \mathbf{x}) \approx \frac{4GM}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \partial_0 z^i\big|_{\tau} \tag{0.19}$$

$$\overline{h}^{00}(t, \mathbf{x}) \approx \frac{4GM}{\|\mathbf{x} - \mathbf{z}(t)\|} \tag{0.20}$$

In particular for $i = j = x$:

$$\overline{h}_{xx}(t, \mathbf{x}) = \overline{h}^{xx}(t, \mathbf{x}) \approx \frac{2GM}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \partial_{00}(x^2)\big|_{\tau} \tag{0.21}$$

The perturbation for two colliding objects

As ODE (0.12) is linear, solutions for more than one *sources* turn out to be the sums of individual solutions. For the two objects[2] calculated in (1) one obtains:

$$\overline{h}_{xx}(t, 0, R, 0) \overset{(0.21)}{\approx} 4GM \cdot \frac{\partial_{00} x^2\big|_{\tau}}{\sqrt{R^2 + x^2(t)}} = \frac{4GM}{9\sqrt{R^2 + x^2(t)}} \cdot \left(\frac{9M}{\tau(t)}\right)^{\frac{2}{3}}$$

$$= \frac{4GM}{9\sqrt{R^2 + x^2(t)}} \cdot \left(\frac{9M}{t - \sqrt{R^2 + x^2(t)}}\right)^{\frac{2}{3}}$$

[2] Assuming low mass, such that $\dot{x} \ll 1$.

As $z^2 = z^3 = 0$ for both objects, obviously $\overline{h}^{ij} = 0$ for all $(i,j) \neq (1,1)$. Thus, the perturbation $\overline{h}_{\mu\nu}$ has trace

$$\overline{h} := \overline{h}^{\mu}{}_{\mu} = \eta_{00}\overline{h}^{00} + \eta_{11}\overline{h}^{xx} \approx \frac{4GM}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \left[\partial_{00}(x^2)\big|_{\tau} - 2\right]$$

With

$$h^{\mu\nu} = \overline{h}^{\mu\nu} + \frac{\eta^{\mu\nu}}{2}h \overset{\overline{h}=-h}{=} \overline{h}^{\mu\nu} - \frac{\eta^{\mu\nu}}{2}\overline{h}$$

it follows

$$h_{xx}(t,\mathbf{x}) = h^{xx}(t,\mathbf{x}) \approx \frac{2GM}{\|\mathbf{x} - \mathbf{z}(t)\|} \cdot \partial_{00}(x^2)\big|_{\tau} + \frac{4GM}{\|\mathbf{x} - \mathbf{z}(t)\|}$$

$$= \frac{2GM}{\sqrt{R^2 + x^2(t)}} \cdot \left[\frac{1}{9}\left(\frac{9M}{t - \sqrt{R^2 + x^2(t)}}\right)^{\frac{2}{3}} + 2\right] \tag{0.22}$$

Problem 03

As was seen in problem (02), the Lorenz gauge $\partial_\mu \overline{h}^{\mu\nu}$ leads to the field equations

$$\Box\overline{h}^{\mu\nu} = -16\pi G T^{\mu\nu} + \mathcal{O}(h^2) \tag{0.23}$$

whereas

$$\overline{h}^{\mu\nu} := h^{\mu\nu} - \frac{\eta^{\mu\nu}}{2}h$$

Thus the equations of motion $\nabla_\nu T^{\mu\nu} = 0$ imply

$$0 \overset{(0.23)}{=} \nabla_\nu \Box\overline{h}^{\mu\nu} + \mathcal{O}(h^2) \overset{\Box = g^{\alpha\beta}\partial_{\alpha\beta}}{=} \partial_\nu \eta^{\alpha\beta}\partial_{\alpha\beta}\overline{h}^{\mu\nu} + \partial_\nu h^{\alpha\beta}\partial_{\alpha\beta}\Box\overline{h}^{\mu\nu} + \Gamma^{\mu}_{\nu\lambda}\Box\overline{h}^{\lambda\nu} + \Gamma^{\nu}_{\nu\lambda}\Box\overline{h}^{\mu\lambda} + \mathcal{O}(h^2)$$

$$= \eta^{\alpha\beta}\partial_{\alpha\beta}\underbrace{\partial_\nu\overline{h}^{\mu\nu}}_{0} + \underbrace{\partial_\nu h^{\alpha\beta}\partial_{\alpha\beta}\Box\overline{h}^{\mu\nu} + \Gamma^{\mu}_{\nu\lambda}\Box\overline{h}^{\lambda\nu} + \Gamma^{\nu}_{\nu\lambda}\Box\overline{h}^{\mu\lambda} + \mathcal{O}(h^2)}_{\substack{\in\mathcal{O}(h^2) \\ \text{as } \Gamma\in\mathcal{O}(h)}} = 0 + \mathcal{O}(h^2)$$

which is true only up to first order in h.