General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 11

February 04, 2010

Problem 01 (Carroll, Problem 5.2)

Consider a perfect fluid in a static, circularly symmetric (2+1)-dimensional spacetime, equivalently, a cylindrical configuration in $(3+1)$-dimensions with perfect rotational symmetry.

(a) Derive the analogue of the Tolman-Oppenheimer-Volkoff (TOV) equation for $(2+1)$ dimensions.

(b) Show that the vacuum solution can be written as

$$ds^2 = -dt^2 + \frac{1}{1-8GM}dr^2 + r^2 d\vartheta^2$$

Here M is a constant.

(c) Show that another way to write the same solution is

$$ds^2 = -d\tau^2 + d\xi^2 + \xi^2 \; d\varphi^2$$

where $\varphi \in [0, 2\pi(1-8GM)^{1/2}]$.

(d) Solve the $(2+1)$ TOV equation for a constant density star. Find $p(r)$ and solve for the metric.

(e) Solve the $(2+1)$ TOV equation for a star with equation of state $p = \varkappa\rho^{3/2}$. Find $p(r)$ and solve for the the metric.

(f) Find the mass $M(R) := \int_0^{2\pi}\int_0^R \rho \; dr \; d\vartheta$ and the proper mass $\overline{M}(R) := \int_0^{2\pi}\int_0^R \rho\sqrt{-g} \; dr \; d\vartheta$ for the solutions in parts (d) and (e).