General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 11 - Solutions

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Preface: Metric on sub-manifolds

Let (M,g) be an n-dimensional semi-Riemann manifold with coordinates $x^1,..,x^n$. Let N be the $(n-1)$-dimensional manifold defined by

$$N := \{x^n : \text{const}\}$$

equipped with the limitation $\widetilde{g}$ of the metric g on TN. Let g be of the Form

$$g = \widetilde{g} + g_{nn} \cdot dx^n dx^n$$

Let from now on latin indices run from 1 to $(n-1)$, greek indices run from 1 to n. Then

1. The Christoffel-Symbols $\widetilde{\Gamma}^k_{ij}$ on N are same as Γ^k_{ij} on M.

2. The Riemann-Tensor $\widetilde{R}$ on N is given by

$$\widetilde{R}^a{}_{bij} = R^a{}_{bij} + \Gamma^n_{ib}\Gamma^a_{jn} - \Gamma^n_{jb}\Gamma^a_{in}$$

   with R being the Riemann-Tensor on M (Note: no summation over n!).

3. The Ricci-Tensor $\widetilde{R}$ on N is given by

$$\widetilde{R}_{ij} = R_{ij} - R^n{}_{inj} + \Gamma^n_{ai}\Gamma^a_{jn} - \Gamma^n_{ji}\Gamma^a_{an}$$

   (Note: no summation over n!)

4. The Ricci-scalar $\widetilde{R}$ on N is given by

$$\widetilde{R} = R - g^{nn}R_{nn} - g^{ib}R^n{}_{bni} + g^{ib}\Gamma^n_{ab}\Gamma^a_{in} - g^{ib}\Gamma^n_{ib}\Gamma^a_{an}$$

   (Note: no summation over n!)

Proof:

1.

$$\widetilde{\Gamma}^k_{ij} = \frac{\widetilde{g}^{kr}}{2}\left(\partial_i\widetilde{g}_{rj} + \partial_j\widetilde{g}_{ri} - \partial_r\widetilde{g}_{ij}\right) = \frac{g^{kr}}{2}\left(\partial_i g_{rj} + \partial_j g_{ri} - \partial_r g_{ij}\right) = \frac{\overbrace{g^{k\mu}}^{\substack{0\\ \text{for}\\ \mu=n}}}{2}\left(\partial_i g_{\mu j} + \partial_j g_{\mu i} - \partial_\mu g_{ij}\right) = \Gamma^k_{ij}$$

2.

$$\widetilde{R}^a{}_{bij} \overset{(1)}{=} \partial_i\Gamma^a_{jb} - \partial_j\Gamma^a_{ib} + \Gamma^l_{jb}\Gamma^a_{il} - \Gamma^l_{ib}\Gamma^a_{jl} = \partial_i\Gamma^a_{jb} - \partial_j\Gamma^a_{ib} + \left(\Gamma^\lambda_{jb}\Gamma^a_{i\lambda} - \Gamma^n_{jb}\Gamma^a_{in}\right) - \left(\Gamma^\lambda_{ib}\Gamma^a_{j\lambda} - \Gamma^n_{ib}\Gamma^a_{jn}\right)$$

$$= R^a{}_{bij} + \Gamma^n_{ib}\Gamma^a_{jn} - \Gamma^n_{jb}\Gamma^a_{in}$$

3.

$$\widetilde{R}_{ij} = \widetilde{R}^{a}{}_{iaj} \overset{(b)}{=} R^{a}{}_{iaj} + \Gamma^{n}_{ai}\Gamma^{a}_{jn} - \Gamma^{n}_{ji}\Gamma^{a}_{an} = (R^{\alpha}{}_{i\alpha j} - R^{n}{}_{inj}) + \Gamma^{n}_{ai}\Gamma^{a}_{jn} - \Gamma^{n}_{ji}\Gamma^{a}_{an}$$

$$= R_{ij} - R^{n}{}_{inj} + \Gamma^{n}_{ai}\Gamma^{a}_{jn} - \Gamma^{n}_{ji}\Gamma^{a}_{an}$$

4.

$$\widetilde{R} = \widetilde{g}^{ib}\widetilde{R}_{ib} = g^{ib}\widetilde{R}_{bi} \overset{(c)}{=} g^{ib}R_{bi} - g^{ib}R^{n}{}_{bni} + g^{ib}\Gamma^{n}_{ab}\Gamma^{a}_{in} - g^{ib}\Gamma^{n}_{ib}\Gamma^{a}_{an}$$

$$= \overbrace{g^{i\beta}}^{\substack{0\\ \text{for}\\ \beta=n}} \cdot R_{\beta i} - g^{ib}R^{n}{}_{bni} + g^{ib}\Gamma^{n}_{ab}\Gamma^{a}_{in} - g^{ib}\Gamma^{n}_{ib}\Gamma^{a}_{an}$$

$$= \Big(g^{\gamma\beta}R_{\beta\gamma} - \underbrace{g^{n\beta}}_{\substack{0\\ \text{for}\\ \beta\neq n}} R_{\beta n}\Big) - g^{ib}R^{n}{}_{bni} + g^{ib}\Gamma^{n}_{ab}\Gamma^{a}_{in} - g^{ib}\Gamma^{n}_{ib}\Gamma^{a}_{an}$$

$$= R - g^{nn}R_{nn} - g^{ib}R^{n}{}_{bni} + g^{ib}\Gamma^{n}_{ab}\Gamma^{a}_{in} - g^{ib}\Gamma^{n}_{ib}\Gamma^{a}_{an}$$

Problem 01

Ricci- & Einstein Tensor

Starting with the general form of a 2-dimensional, static, circularly symmetric metric

$$\widetilde{g} = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\vartheta^2$$

we notice that this is actually the limitation of the 4-dimensional metric

$$g = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2\ d\vartheta^2 + r^2 \sin^2\vartheta\ d\varphi^2$$

on the sub-manifold $\{\varphi : \text{const}\}$. Using the above results we get the 3-dimensional, non-trivial components of the Ricci-Tensor

$$\widetilde{R}_{tt} = R_{tt} - R^{\varphi}{}_{t\varphi t} = R_{tt} - \frac{e^{2(\alpha-\beta)}}{r} \cdot \partial_r\alpha = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{\partial_r\alpha}{r}\right]$$

$$\widetilde{R}_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{\partial_r\beta}{r}$$

$$\widetilde{R}_{\vartheta\vartheta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha)\right]$$

where use has been made of the results in problem set # 8 and

$$\underbrace{R^{\mu}{}_{\nu\mu\varkappa}}_{\substack{\text{no}\\ \text{summation}}} = \underbrace{g^{\mu\lambda}R_{\lambda\nu\mu\varkappa}}_{\substack{\text{summation}\\ \text{over } \lambda}} \overset{\substack{g\\ \text{diagonal}}}{=} \underbrace{g^{\mu\mu}R_{\mu\nu\mu\varkappa}}_{\substack{\text{no}\\ \text{summation}}} = \underbrace{g^{\nu\nu}R_{\nu\mu\varkappa\mu} \cdot \frac{g^{\mu\mu}}{g^{\nu\nu}}}_{\substack{\text{no}\\ \text{summation}}} \overset{\substack{g\\ \text{diagonal}}}{=} \underbrace{g^{\nu\lambda}R_{\lambda\mu\varkappa\mu} \cdot \frac{g^{\mu\mu}}{g^{\nu\nu}}}_{\substack{\text{summation}\\ \text{over } \lambda}} = \underbrace{R^{\nu}{}_{\mu\varkappa\mu} \cdot \frac{g^{\mu\mu}}{g^{\nu\nu}}}_{\substack{\text{no}\\ \text{summation}}}$$

for diagonal metrics, in particular

$$\underbrace{R^{\varphi}{}_{i\varphi k}}_{\substack{\text{no}\\ \text{summation}}} = \underbrace{R^{i}{}_{\varphi k\varphi} \cdot \frac{g^{\varphi\varphi}}{g^{ii}}}_{\substack{\text{no}\\ \text{summation}}}$$

Consequently we obtain the Ricci-scalar

$$\widetilde{R} = \widetilde{g}^{ij}\widetilde{R}_{ij} = 2e^{-2\beta}\left[-\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{1}{r}\left(\partial_r\beta - \partial_r\alpha\right)\right]$$

Finally we obtain the non-trivial components of the Einstein-Tensor

$$\widetilde{G}_{tt} = \widetilde{R}_{tt} - \frac{\widetilde{g}_{tt}}{2}\widetilde{R} = \frac{e^{2(\alpha-\beta)}}{r} \cdot \partial_r \beta$$

$$\widetilde{G}_{rr} = \widetilde{R}_{rr} - \frac{\widetilde{g}_{rr}}{2}\widetilde{R} = \frac{\partial_r \alpha}{r}$$

$$\widetilde{G}_{\vartheta\vartheta} = \widetilde{R}_{\vartheta\vartheta} - \frac{\widetilde{g}_{\vartheta\vartheta}}{2}\widetilde{R} = e^{-2\beta}\left[r\left(\partial_r\beta - \partial_r\alpha\right) + r^2\partial_r^2\alpha + r^2(\partial_r\alpha)^2 - r^2\partial_r\alpha\partial_r\beta\right]$$

Perfect, rotational-symmetric fluid

The energy-momentum tensor for a perfect fluid is given by

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + p g_{\mu\nu} \tag{0.1}$$

with U as the velocity vector field for the fluid. We introduce co-flowing coordinates[1], so that

$$U^\mu = \left(\frac{1}{\sqrt{|g_{tt}|}},\ 0,\ 0\right) \ \rightsquigarrow\ U_\mu = \left(-\sqrt{|g_{tt}|},\ 0,\ 0\right)$$

and consequently

$$T_{\mu\nu} = (p+\rho)\cdot e^{2\alpha}\cdot\delta_{\mu t}\delta_{\nu t} + p g_{\mu\nu}$$

In particular, the only non-trivial components of $T^{\mu\nu}$ are given by

$$T^{tt} = \rho\cdot e^{-2\alpha} \quad , \quad T^{rr} = p\cdot e^{-2\beta} \quad , \quad T^{\vartheta\vartheta} = \frac{p}{r^2}$$

The Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{0.2}$$

imply

$$\partial_r\beta(r) = 8\pi r\rho(r)\cdot e^{2\beta(r)} \tag{0.3}$$

and

$$\partial_r\alpha(r) = 8\pi r p(r)\cdot e^{2\beta(r)} \tag{0.4}$$

The ODE (0.3) is equivalent to

$$\boxed{\partial_r m(r) = 2\pi r\rho(r)} \tag{0.5}$$

whereas

$$m(r) := \frac{1}{8}\left[1 - e^{-2\beta(r)}\right] \tag{0.6}$$

with boundary condition $m(0) = 0$ (that is, $e^{2\beta(0)} = 1$). One directly obtains

$$\boxed{m(r) = 2\pi\int_0^r r\rho(r)\,dr} \tag{0.7}$$

Conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ implies

$$0 = \nabla_\mu T^{\mu r} = \partial_r T^{rr} + \left(\Gamma^t_{tr} + \Gamma^\vartheta_{\vartheta r} + \Gamma^r_{rr}\right)T^{rr} + \Gamma^r_{tt}T^{tt} + \Gamma^r_{rr}T^{rr} + \Gamma^r_{\vartheta\vartheta}T^{\vartheta\vartheta}$$

$$= e^{-2\beta}\left[(p+\rho)\partial_r\alpha + \partial_r p\right]$$

and together with (0.4) & (0.6) the TOV equation

$$\boxed{\frac{dp}{dr} = -(p+\rho)\cdot\frac{8\pi r p(r)}{1 - 8m(r)}} \tag{0.8}$$

[1] Note that these preserve rotational symmetry & time-independence.

Vacuum

In case of vacuum, that is

$$\widetilde{R}_{tt} = \widetilde{R}_{rr} = \widetilde{R}_{\vartheta\vartheta} = 0$$

one obtains

$$0 = \widetilde{R}_{tt} + e^{2(\alpha-\beta)}\widetilde{R}_{rr} = \frac{1}{r}e^{2(\alpha-\beta)}\left[\partial_r\alpha + \partial_r\beta\right]$$

that is

$$\partial_r\alpha = -\partial_r\beta \tag{0.9}$$

Using $\widetilde{R}_{\vartheta\vartheta} = 0$ one gets

$$\partial_r\alpha = \partial_r\beta$$

and thus together with (0.9)

$$\partial_r\alpha = 0 = \partial_r\beta \tag{0.10}$$

or equivalently

$$\alpha = A : \text{const} \quad , \quad \beta = B : \text{const}$$

Consequently

$$g = -e^{2A}dt^2 + e^{2B}dr^2 + r^2 d\vartheta^2 \tag{0.11}$$

By substituting

$$\widetilde{t} = \widetilde{t}(t) := e^A t \quad , \quad M := \frac{1-e^{-2B}}{8}$$

one obtains

$$\boxed{g = -d\widetilde{t}^2 + \frac{1}{1-8M}dr^2 + r^2 d\vartheta^2} \tag{0.12}$$

Further substituting

$$\tau := \widetilde{t} \quad , \quad \xi = \xi(r) := \frac{r}{\sqrt{1-8M}} \quad , \quad \varphi := \sqrt{1-8M}\cdot\vartheta$$

leads to

$$\boxed{g = -d\tau^2 + d\xi^2 + \xi^2 d\varphi^2} \tag{0.13}$$

whereas

$$\varphi \in \left[0,\ 2\pi\sqrt{1-8M}\right]$$

Constant density EOS

Let $\rho \equiv \rho_0$ be constant, then

$$\boxed{m(r) \overset{(0.7)}{=} \pi r^2 \rho_0} \tag{0.14}$$

and the TOV equation (0.8) takes the form

$$\frac{dp}{dr} = -(p+\rho_0)\cdot\frac{8\pi r p}{1-8\pi\rho_0 r^2}$$

which solves as

$$\frac{\ln p}{\rho_0} - \frac{\ln(p+\rho_0)}{\rho_0} = \int\frac{dp}{(p+\rho_0)p} = -\int\frac{8\pi r}{1-8\pi\rho_0 r^2}\,dr = \frac{1}{\rho_0}\ln\left|1-8\pi\rho_0 r^2\right|^{\frac{1}{2}} + \text{const}$$

to give

$$p(r) = \rho_0 C \cdot \frac{\left|1 - 8\pi\rho_0 r^2\right|^{\frac{1}{2}}}{1 - \left|1 - 8\pi\rho_0 r^2\right|^{\frac{1}{2}}} \quad , \quad C : \text{const} \tag{0.15}$$

Furthermore, inserting m and p into (0.4) yields

$$\partial_r \alpha = 8\pi\rho_0 r C \cdot \frac{\left|1 - 8\pi\rho_0 r^2\right|^{-\frac{1}{2}}}{1 - \left|1 - 8\pi\rho_0 r^2\right|^{\frac{1}{2}}}$$

which implies

$$\alpha(r) = C \ln\left\{1 - \left|1 - 8\pi\rho_0 r^2\right|^{\frac{1}{2}}\right\} + 2A \quad , \quad A > 0 \tag{0.16}$$

As a simple time-scaling $t \mapsto e^A t$ eliminates the constant A, we w.l.o.g. assume $A = 0$ and obtain the metric

$$\boxed{g = -\left[1 - \sqrt{1 - 8\pi\rho_0 r^2}\right]^{2C} dt^2 + \frac{1}{1 - 8\pi\rho_0 r^2}\, dr^2 + r^2 d\vartheta^2} \tag{0.17}$$

The proper mass $\overline{M}(R)$ is in particular given by

$$\overline{M}(R) = 2\pi\rho_0 \int_0^R \sqrt{|g_s|}\, dr = 2\pi\rho_0 \int_0^R \frac{r\, dr}{\sqrt{1 - 8\pi\rho_0 r^2}} = \frac{1}{4}\left[1 - \sqrt{1 - 8\pi\rho_0 R^2}\right]$$

with $g_s = g_{rr} dr^2 + g_{\vartheta\vartheta} d\vartheta^2$ as the spatial part of g.

Simple EOS

Let

$$p = \varkappa \rho^{\frac{3}{2}} \tag{0.18}$$

be the equation of state for the considered fluid. Then (0.8) takes the form

$$\underbrace{\frac{1}{\varkappa\rho^{\frac{3}{2}} + \rho} \frac{d\rho}{dr}}_{\frac{d}{dr}\left[\ln\rho - 2\ln(\varkappa\sqrt{\rho}+1)\right]} = -\frac{16\pi r\rho}{3(1 - 8m(r))} \overset{(0.5)}{=} \frac{1}{3}\frac{d}{dr} \ln|1 - 8m(r)|$$

from which

$$\frac{\rho(r)}{(\varkappa\sqrt{\rho(r)} + 1)^2} = C \cdot |1 - 8m(r)|^{\frac{1}{3}} \quad , \quad C > 0 \tag{0.19}$$

follows. Solving for m and differentiation with respect to r, yields

$$\frac{\rho}{\left(\varkappa\sqrt{\rho} + 1\right)^7} \frac{d\rho}{dr} = -\frac{16}{3} C^3 \pi r$$

which solves as

$$\boxed{\frac{1 + 6\varkappa\sqrt{\rho} + 16\varkappa^2\rho + 20\varkappa^3\rho^{\frac{3}{2}}}{10\varkappa^4\left(1 + \varkappa\sqrt{\rho}\right)^6} = -8C^3\pi r^2 + A \quad , \quad A : \text{const}} \tag{0.20}$$

Consistency with (0.19) (in particular at $r = 0$) implies

$$A = \frac{1 + 6\varkappa\sqrt{\rho_0} + 16\varkappa^2\rho_0 + 20\varkappa^3\rho_0^{\frac{3}{2}}}{10\varkappa^4\left(1 + \varkappa\sqrt{\rho_0}\right)^6} \quad , \quad \rho_0 := \rho(0) \overset{(0.19)}{=} \frac{C}{\left(1 - \varkappa\sqrt{C}\right)^2}$$

Relations (0.6) & (0.19) imply

$$e^{2\beta} = \frac{1}{1-8m} = C^3 \cdot \frac{\left(\varkappa\sqrt{\rho}+1\right)^6}{\rho^3} \tag{0.21}$$

Solving (0.19) for ρ yields

$$\rho(r) = \frac{|1-8m(r)|^{\frac{1}{3}}\, C}{\left[1-\varkappa\,|1-8m(r)|^{\frac{1}{6}}\,\sqrt{C}\right]^2} \tag{0.22}$$

and consequently

$$\boxed{p(r) = \frac{\varkappa \cdot |1-8m(r)|^{\frac{1}{2}}\, C^{\frac{3}{2}}}{\left[1-\varkappa\,|1-8m(r)|^{\frac{1}{6}}\,\sqrt{C}\right]^3}} \tag{0.23}$$

Furthermore, relations (0.4), (0.6) & (0.18) imply

$$\partial_r\alpha = \frac{8\varkappa\pi r\rho^{\frac{3}{2}}}{1-8m} \overset{(0.22)}{=} \frac{8\varkappa\pi r\rho\,|1-8m|^{-\frac{5}{6}}\,\sqrt{C}}{\left[1-\varkappa\,|1-8m|^{\frac{1}{6}}\,\sqrt{C}\right]^2} \overset{(0.5)}{=} -3\frac{d}{dr}\left\{\frac{1}{1-\varkappa\,|1-8m|^{\frac{1}{6}}\,\sqrt{C}}\right\}$$

and thus

$$\alpha(r) = \frac{-3}{1-\varkappa\,|1-8m(r)|^{\frac{1}{6}}\,\sqrt{C}} + B \quad , \quad B : \text{const} \tag{0.24}$$

As B can be eliminated by a simple time-scaling $t \mapsto e^B t$, we w.l.o.g. assume $B = 0$ and obtain the metric

$$\boxed{g = \exp\left[\frac{-6}{1-\varkappa\,|1-8m(r)|^{\frac{1}{6}}\,\sqrt{C}}\right]\, dt^2 + \frac{dr^2}{1-8m(r)} + r^2 d\vartheta^2} \tag{0.25}$$