General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 10

January 28, 2010

Problem 01

Derive a first order differential equation for the trajectory (r as a function of φ) for equatorial orbits in the Schwarzschild geometry.

Problem 02 (Carroll, Problem 5.3)

Consider a particle (not necessary on a geodesic) that has fallen inside the event horizon, $r < 2GM$. Use the ordinary Schwarzschild coordinates $(t, r, \vartheta, \varphi)$. Show that the radial coordinate must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}$$

Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM$ to $r = 0$. Express this in seconds for a black hole with mass measured in solar masses.

Problem 03 (Carroll, Problem 5.5)

Consider a comoving observer sitting at constant spatial coordinates $(r_0, \vartheta_0, \varphi_0)$, around a Schwarzschild black hole of mass M. The observer drops a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength λ_{em} (in the beacon rest frame).

(a) Calculate the coordinate speed dr/dt of the beacon, as a function of r.

(b) Calculate the proper speed of the beacon. That is, imagine there is a comoving observer at fixed r, with a locally inertial coordinate system set up as the beacon passes by, and calculate the speed as measured by the comoving observer. What is it at $r = 2GM$?

(c) Calculate the wavelength λ_{obs}, measured by the observer at r_0, as a function of the radius r_{em} at which the radiation was emitted.

(d) Calculate the time t_{obs} at which a beam emitted by the beacon at radius r_{em} will be observed at r_0.

(e) Show that at late times, the redshift grows exponentially: $\lambda_{\text{obs}}/\lambda_{\text{em}} \propto e^{t_{\text{obs}}/T}$. Give an expression for the time constant T in terms of the black hole mass M.