General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 10 - Solutions

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February 10, 2010

Problem 01

The equations of motion (EOM) of *free* particles in the Schwarzschild metric can be described by the Lagrangian

$$\mathcal{L}(x,\dot{x}) = g(\dot{x},\dot{x}) = -\left(1-\frac{2M}{r}\right)\cdot\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\cdot\dot{r}^2 + r^2\cdot\dot{\vartheta}^2 + r^2\sin^2\vartheta\cdot\dot{\varphi}^2 \tag{0.1}$$

and the Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0 \tag{0.2}$$

whereas for the orbit we assume the curve parameter to be the proper time τ:

$$g(\dot{x},\dot{x}) \equiv -s : \text{const} \tag{0.3}$$

with $s=0$ for photons and $s=1$ for massive particles. Equations (0.2) lead to

$$0 = \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \underbrace{\frac{\partial\mathcal{L}}{\partial\varphi}}_{0} = \frac{d}{d\tau}\left(2r^2\sin^2\vartheta\cdot\dot{\varphi}\right) \quad\Rightarrow\quad r^2\sin^2\vartheta\cdot\dot{\varphi} = L : \text{const} \tag{0.4}$$

and

$$0 = \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{t}} - \underbrace{\frac{\partial\mathcal{L}}{\partial t}}_{0} = -\frac{d}{d\tau}\left[2\left(1-\frac{2M}{r}\right)\cdot\dot{t}\right] \quad\Rightarrow\quad \left(1-\frac{2M}{r}\right)\cdot\dot{t} = E : \text{const} \tag{0.5}$$

In the special case of equatorial orbits, that is $\vartheta\equiv\pi/2$, eq. (0.3) takes the form

$$-\left(1-\frac{2M}{r}\right)\cdot\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\cdot\dot{r}^2 + r^2\cdot\dot{\varphi}^2 + s = 0$$

which together with (0.4) & (0.5) leads to

$$\dot{r}^2 = E^2 - s\cdot\left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)\cdot\frac{L^2}{r^2} \tag{0.6}$$

Finally

$$\boxed{\frac{dr}{d\varphi} = \frac{dr}{d\tau}\cdot\frac{d\tau}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} \overset{(0.4)\&(0.6)}{=} \frac{r}{L}\cdot\sqrt{r^2E^2 - rs\cdot(r-2M) - \left(1-\frac{2M}{r}\right)\cdot L^2}} \tag{0.7}$$

whereas we w.l.o.g. we assumed $\frac{dr}{d\varphi}\geq 0$. The constants E^2, L can be interpreted as *total energy* and *angular* momentum of the particle respectively.

Problem 02

Let $\dot{x} = (\dot{t}, \dot{r}, \dot{\vartheta}, \dot{\varphi})$ be the velocity of the particle and τ be the proper time of the particle. Then causality implies

$$-1 = g(\dot{x},\dot{x}) = \left(\frac{2GM}{r} - 1\right) \cdot \dot{t}^2 - \left(\frac{2GM}{r} - 1\right)^{-1} \cdot \dot{r}^2 + r^2 \cdot \dot{\vartheta}^2 + r^2 \sin^2\vartheta \cdot \dot{\varphi}^2$$

that is

$$\dot{r}^2 = \left(\frac{2GM}{r} - 1\right) + \underbrace{\left(\frac{2GM}{r} - 1\right)^2 \cdot \dot{t}^2}_{\geq 0} + \underbrace{\left(\frac{2GM}{r} - 1\right) \cdot r^2 \cdot \dot{\vartheta}^2}_{\geq 0} + \underbrace{\left(\frac{2GM}{r} - 1\right) \cdot r^2 \sin^2\vartheta \cdot \dot{\varphi}^2}_{\geq 0}$$

$$\geq \left(\frac{2GM}{r} - 1\right)$$

and thus

$$\boxed{\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}} \tag{0.8}$$

From (0.8) one can see, that if $r(\tau_0) < 2GM,\ \dot{r}(\tau_0) < 0$ at some point τ_0 ($\leftrightarrow$ particle *falls* into the black hole), then[1] $\dot{r}(\tau) \leq \dot{r}(\tau_0)\quad \forall\ \tau \geq \tau_0$ and the particle inevitably falls into the origin $r = 0$. Thus the life-time T of a particle within the event-horizon is given by

$$0 = 2GM + \int_0^T \underbrace{\dot{r(\tau)}}_{<0}\ d\tau = 2GM - \int_0^T |\dot{r}|\ d\tau$$

In particular, T decreases as $|\dot{r}|$ increases (for varying trajectories). From the above calculations, one can see that $|\dot{r}|$ is minimal, if $\dot{t} = \dot{\vartheta} = \dot{\varphi} = 0$ and thus T maximal for the trajectory

$$-\dot{r} = |\dot{r}| = \sqrt{\frac{2GM}{r} - 1} \tag{0.9}$$

Integrating the ODE (0.9) leads to

$$T_{\max} = \int_0^T d\tau = \int_0^{2GM} \frac{dr}{\sqrt{\frac{2GM}{r} - 1}} = r \cdot \sqrt{\frac{2GM}{r} - 1} - M \cdot \arctan\left[\frac{(M-r)}{2M-r} \cdot \sqrt{\frac{2GM}{r} - 1}\right]\Bigg|_0^{2GM} = \pi M \tag{0.10}$$

that is[2]

$$\boxed{T_{\max} \approx 1.55 \times 10^{-5}\ \mathrm{s} \cdot \frac{M}{M_\odot}} \tag{0.11}$$

Problem 03

Let $(t_0, r_0, \vartheta_0, \varphi_0)$ be the start position of the beacon trajectory $x(\tau)$, $r_0 > 2GM$ and τ its proper time starting at drop-point.

(a) In analogy to (0.5) in problem 01 the beacon-trajectory satisfies

$$\left(1 - \frac{2M}{r}\right) \cdot \dot{t} = E : \mathrm{const} \tag{0.12}$$

[1] This can be seen as follows: $\dot{r}$ can not be positive after τ_0 since then there would exist some $\tau_1 > \tau_0$ with $\dot{r}(\tau_1) = 0$ and $\dot{r}(\tau) \leq 0\ \ \forall\ \tau_0 \leq \tau \leq \tau_1$ But this means $r(\tau_1) = 2GM$, which is in contradiction to $r(\tau_0) < 2GM$.
Thus $\dot{r}(\tau) < 0\ \ \forall\ \tau \geq \tau_0$, which implies $r(\tau) \leq r(\tau_0)\ \ \forall\ \tau \geq \tau_0$ and due to (0.8) thus $\dot{r}(\tau) \leq \dot{r}(\tau_0)\ \ \forall\ \tau \geq \tau_0$.

[2] $\mathbb{M}_\odot \approx 4.95 \times 10^{-6}$ s.

Dropping the beacon implies $\dot{r}(0) = \dot{\vartheta}(0) = \dot{\varphi}(0) = 0$ so that

$$-1 = g(\dot{x}(0), \dot{x}(0)) = -\left(1 - \frac{2GM}{r_0}\right) \cdot \dot{t}^2(0)$$

which implies

$$\dot{t}(0) = \frac{1}{\sqrt{1 - \frac{2GM}{r_0}}} \tag{0.13}$$

and thus

$$E \overset{(0.12)}{=} \sqrt{1 - \frac{2GM}{r_0}} \tag{0.14}$$

Condition $-1 = g(\dot{x}, \dot{x})$ takes for radial trajectories the form

$$-1 = -\left(1 - \frac{2GM}{r}\right) \cdot \dot{t}^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \cdot \dot{r}^2$$

which together with (0.12) and (0.14) implies

$$\dot{r}^2 = E^2 - \left(1 - \frac{2GM}{r}\right) = \frac{2GM}{r} - \frac{2GM}{r_0} \tag{0.15}$$

and thus

$$\boxed{\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = -\left(1 - \frac{2GM}{r}\right) \cdot \sqrt{\frac{2GM}{r} \cdot \frac{r_0 - r}{r_0 - 2GM}}} \tag{0.16}$$

(b) Equation (0.15) leads to the proper speed

$$\boxed{\dot{r} = \frac{dr}{d\tau} = -\sqrt{2GM \cdot \left(\frac{1}{r} - \frac{1}{r_0}\right)}} \tag{0.17}$$

In particular

$$\boxed{\left.\frac{dr}{d\tau}\right|_{r=2GM} = -\sqrt{\frac{r_0 - 2GM}{r_0}}} \tag{0.18}$$

Furthermore

$$\boxed{\dot{t} = \frac{dt}{d\tau} = \left(\frac{dr}{dt}\right)^{-1} \frac{dr}{d\tau} = \left(1 - \frac{2GM}{r}\right)^{-1} \cdot \sqrt{1 - \frac{2GM}{r_0}}} \tag{0.19}$$

(c) The photons $x_p(\lambda)$ transmitted by the beacon, reaching the observer, are those transmitted along the previously followed world line in positive r-direction, thus

$$\dot{x}_p = \dot{t}_p \partial_t + \dot{r}_p \partial_r \overset{g(\dot{x}_p, \dot{x}_p)=0}{=} \dot{t}_p \partial_t + \sqrt{\left|\frac{g_{tt}}{g_{rr}}\right|} \cdot \dot{t}_p \partial_r \tag{0.20}$$

As the considered observer rests at r_0, we may use (0.27) and obtain

$$\lambda_{\text{obs}} = \lambda_{\text{em}} \cdot \frac{\sqrt{|g_{tt}|\,|_{\text{obs}}}}{g_{tt}|_{\text{em}}} \cdot \frac{g(\dot{x}_p, \dot{x})}{\dot{t}_p|_{\text{em}}} = \lambda_{\text{em}} \cdot \frac{\sqrt{|g_{tt}|\,|_{\text{obs}}}}{g_{tt}|_{\text{em}}} \cdot \frac{1}{\dot{t}_p|_{\text{em}}} \cdot \left[g_{tt} \dot{t}_p \dot{t} + g_{rr} \dot{r}_p \dot{r}\right]\Big|_{\text{em}}$$

$$\overset{(0.20)}{=} \lambda_{\text{em}} \cdot \frac{\sqrt{|g_{tt}|\,|_{\text{obs}}}}{g_{tt}|_{\text{em}}} \cdot \left[g_{tt} \dot{t} + g_{rr} \dot{r} \cdot \sqrt{\left|\frac{g_{tt}}{g_{rr}}\right|}\right]\Bigg|_{\text{em}} \overset{\text{sgn}(g_{rr}) \neq \text{sgn}(g_{tt})}{=} \lambda_{\text{em}} \cdot \sqrt{|g_{tt}|\,|_{\text{obs}}} \cdot \left[\dot{t} - \dot{r} \cdot \sqrt{\left|\frac{g_{rr}}{g_{tt}}\right|}\right]\Bigg|_{\text{em}}$$

$$\overset{g_{tt} = g_{rr}^{-1}}{=} \lambda_{\text{em}} \cdot \sqrt{|g_{tt}|\,|_{\text{obs}}} \cdot \left[\dot{t} - \frac{\dot{r}}{|g_{tt}|}\right]\Bigg|_{\text{em}}$$

Finally, using (0.17) and (0.19) leads to

$$\boxed{\lambda_{\text{obs}} = \lambda_{\text{em}} \cdot \left(1 - \frac{2GM}{r_0}\right)\left(1 - \frac{2GM}{r_{\text{em}}}\right)^{-1} \cdot \left[1 + \sqrt{\frac{2GM(r_0 - r_{\text{em}})}{r_{\text{em}}(r_0 - 2GM)}}\right]} \tag{0.21}$$

In particular $\lambda_{\text{obs}} \overset{r\to 2GM^+}{\longrightarrow} \infty$.

(d) From eq. (0.20) one gets for the photon headed to the observer

$$\frac{dr_p}{dt_p} = \frac{\dot{r}_p}{\dot{t}_p} = \sqrt{\left|\frac{g_{tt}}{g_{rr}}\right|} = 1 - \frac{2GM}{r}$$

Solving this ODE delivers the time t_{obs} needed for the photon to travel from r_{em} to r_0:

$$t_{\text{obs}} = \int_0^{t_{\text{obs}}} dt_p = \int_{r_{\text{em}}}^{r_0} \frac{dr}{(1 - \frac{2GM}{r})} = (r_0 - r_{\text{em}}) + 2GM \ln\left[\frac{r_0 - 2GM}{r_{\text{em}} - 2GM}\right] \tag{0.22}$$

In particular $t_{\text{obs}} \overset{r\to 2GM^+}{\longrightarrow} \infty$.

(e) Since $t_{\text{obs}} \to \infty$ is equivalent to $r_{\text{em}} \to 2GM$, eq. (0.21) takes for late times the form

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \left(1 - \frac{2GM}{r_0}\right)\frac{r_{\text{em}}}{r_{\text{em}} - 2GM} \cdot \left[1 + \underbrace{\sqrt{\frac{2GM(r_0 - r_{\text{em}})}{r_{\text{em}}(r_0 - 2GM)}}}_{1+\mathcal{O}(r_{\text{em}}-2GM)}\right]$$

$$= \left(1 - \frac{2GM}{r_0}\right) \cdot \frac{4GM}{r_{\text{em}} - 2GM} + \mathcal{O}(1) + \mathcal{O}(r_{\text{em}} - 2GM)$$

Solving (0.22) for r_{em} leads to

$$r_{\text{em}} - 2GM = (r_0 - 2GM)\exp\left[\frac{r_0 - r_{\text{em}} - t_{\text{obs}}}{2GM}\right]$$

so that for large t_{obs}

$$\boxed{\frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} \propto \exp\left[\frac{t_{\text{obs}}}{2GM}\right]} \tag{0.23}$$

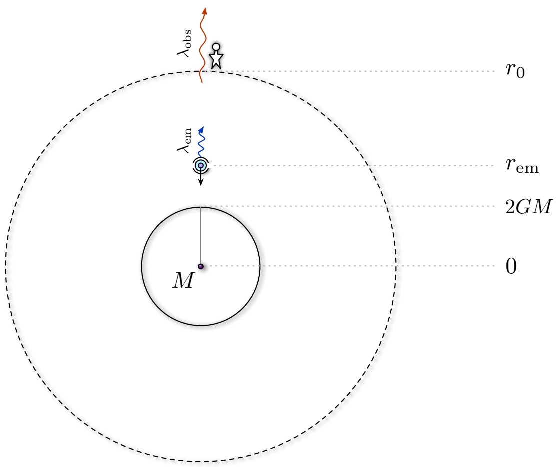

Figure 0.1: Illustration of the beacon falling along a radial trajectory into the black hole.

Notes on the Doppler-effect & redshift

A photon on a (geodesic) world line $x_p(\lambda)$, appears to an observer on a (not necessarily geodesic) world line $x_1(\tau)$ at frequency

$$\nu = \nu_0 \cdot g(\dot{x}_p, \dot{x}_1)$$

with ν_0 being some constant[3]. Keeping in mind that for static metrics[4] *energy* is conserved[5]

$$g_{tt} \cdot \dot{t}_p = E : \text{const}$$

(see proof below) the ratio of frequencies ω_1, ω_2 at which two observers $x_1(\tau_1), x_2(\tau_2)$ observe that same photon is given by

$$\frac{\nu_1}{\nu_2} = \frac{g(\dot{x}_p, \dot{x}_1)}{g(\dot{x}_p, \dot{x}_2)} = \frac{E \cdot \dot{t}_1 + g_s(\dot{\mathbf{x}}_p, \dot{\mathbf{x}}_1)}{E \cdot \dot{t}_2 + g_s(\dot{\mathbf{x}}_p, \dot{\mathbf{x}}_2)} \tag{0.24}$$

with $g_s, \mathbf{x}_p, \mathbf{x}_1, \mathbf{x}_2$ being the spatial parts of the metric and the three world lines respectively. In case of a photon emitted by a beacon x_b at local frequency ν_b, the frequency ν_{obs} for a remote, resting[6] observer x_{obs} is given by

$$\nu_{\text{obs}} = \nu_b \cdot \frac{E \cdot \dot{t}_{\text{obs}}}{g(\dot{x}_p, \dot{x}_b)} \tag{0.25}$$

Through $g(\dot{x}_{\text{obs}}, \dot{x}_{\text{obs}}) = -1$ it follows

$$\dot{t}_{\text{obs}} = \frac{1}{\sqrt{|g_{tt}|}} \tag{0.26}$$

and thus

$$\boxed{\nu_{\text{obs}} = \frac{g_{tt}\big|_b}{\sqrt{|g_{tt}|\,\big|_{\text{obs}}}} \cdot \frac{\nu_b \cdot \dot{t}_p\big|_b}{g(\dot{x}_p, \dot{x}_b)}} \tag{0.27}$$

In case of a resting beacon as well[7] (0.27) simplifies further to

$$\boxed{\nu_{\text{obs}} = \nu_b \cdot \sqrt{\frac{|g_{tt}|\,\big|_b}{|g_{tt}|\,\big|_{\text{obs}}}}} \tag{0.28}$$

(comp. gravitational *redshift*).

On the conservation of energy along geodesics

Let g be such that for some coordinate index $\varkappa$:

$$\partial_\varkappa g_{\mu\nu} = 0 \quad \forall\ \mu, \nu \quad \wedge \quad g = g_{\varkappa\varkappa} dx^\varkappa dx^\varkappa + g_s$$

with $g_s := g\big|_{T\{x^\varkappa:\text{const}\}}$. Then for any curve satisfying the geodesic equation $\nabla_{\dot{x}} \dot{x} = 0$, that is

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma \tag{0.29}$$

the value

$$E := \dot{x}^\varkappa \cdot g_{\varkappa\varkappa}$$

is constant along the curve.

[3]If the photon is *emitted* by the beacon $x_b(\tau)$ at local frequency ν_b, then $\nu_0 = \frac{\nu_b}{g(\dot{x}_p, \dot{x}_b)}$.
[4]Thus $\partial_t g_{\mu\nu} = 0$ and $g = g_{tt} dt^2 + g_s$, with $g_s := g\big|_{T\{t=\text{const}\}}$ being the spatial part.
[5]Thus given by $E = \frac{1}{X^t} \cdot [g(\dot{x}_p, X) - g_s(\dot{\mathbf{x}}_p, \mathbf{X})]$ for any tangent vector $X = X^t \partial_t + \mathbf{X}$.
[6]That is, $\dot{x}_{\text{obs}} = \dot{t}_{\text{obs}} \partial_t$.
[7]That is, $\dot{x}_b = \dot{t}_b \cdot \partial_t$ and $\dot{\mathbf{x}}_b = 0$.

Example: For static metrics g, that is $\partial_t g_{\mu\nu} = 0$ and $g = g_{tt}dt^2 + g_s$ with the spatial part $g_s = g\big|_{T\{t:\text{const}\}}$, the energy

$$E := g_{tt} \cdot \dot{t}$$

is conserved.

Proof: Let $x = x(\tau)$. Then

$$\dot{E} = \ddot{x}^{\varkappa} g_{\varkappa\varkappa} + \dot{x}^{\varkappa} \dot{g}_{\varkappa\varkappa} \overset{(0.29)}{=} -\Gamma^{\varkappa}_{\rho\sigma} \dot{x}^{\rho} \dot{x}^{\sigma} g_{\varkappa\varkappa} + \dot{x}^{\varkappa} \dot{x}^{\mu} \partial_{\mu} g_{\varkappa\varkappa} = -\frac{g^{\varkappa\lambda}}{2} \left(\partial_{\rho} g_{\sigma\lambda} + \partial_{\sigma} g_{\rho\lambda} - \partial_{\lambda} g_{\rho\sigma}\right) \dot{x}^{\rho} \dot{x}^{\sigma} g_{\varkappa\varkappa} + \dot{x}^{\varkappa} \dot{x}^{\mu} \partial_{\mu} g_{\varkappa\varkappa}$$

$$= -\frac{g^{\varkappa\varkappa}}{2} (\partial_{\rho} g_{\sigma\varkappa} + \partial_{\sigma} g_{\rho\varkappa} - \underbrace{\partial_{\varkappa} g_{\rho\sigma}}_{\varkappa}) \dot{x}^{\rho} \dot{x}^{\sigma} g_{\varkappa\varkappa} + \dot{x}^{\varkappa} \dot{x}^{\mu} \partial_{\mu} g_{\varkappa\varkappa} = -\frac{g^{\varkappa\varkappa}}{2} \left(\dot{x}^{\rho} \dot{x}^{\varkappa} \partial_{\rho} g_{\varkappa\varkappa} + \dot{x}^{\varkappa} \dot{x}^{\sigma} \partial_{\sigma} g_{\varkappa\varkappa}\right) g_{\varkappa\varkappa} + \dot{x}^{\varkappa} \dot{x}^{\mu} \partial_{\mu} g_{\varkappa\varkappa}$$

$$= -\underbrace{g^{\varkappa\varkappa} g_{\varkappa\varkappa}}_{1} \dot{x}^{\mu} \dot{x}^{\varkappa} \partial_{\mu} g_{\varkappa\varkappa} + \dot{x}^{\mu} \dot{x}^{\varkappa} \partial_{\mu} g_{\varkappa\varkappa} = 0$$

□