GR: Problem Set #9

1. The Weyl tensor is defined in n dimensions as

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)}g_{\rho[\mu}g_{\nu]\sigma}R.$$

   Show that the Weyl tensor $C^{\mu}{}_{\nu\rho\sigma}$ is left invariant by a conformal transformation.

2. (Carroll, Problem 4.3) The four-dimensional δ-function on a manifold M is defined by

$$\int_M F(x^\mu)\left[\frac{\delta^{(4)}(x^\sigma - y^\sigma)}{\sqrt{-g}}\right]\sqrt{-g}\,dx^4 = F(y^\sigma)\,,$$

   for an arbitrary function $F(x^\mu)$. Meanwhile, the energy-momentum tensor for a pressure-less perfect fluid (dust) is

$$T^{\mu\nu} = \rho\,U^\mu\,U^\nu\,,$$

   where ρ is the energy density and U^μ is the four-velocity. Consider such a fluid that consists of a single particle traveling on a world line $x^\mu(\tau)$, with τ the proper time. The energy-momentum tensor for this fluid is then given by

$$T^{\mu\nu}(y^\sigma) = m\int_M \left[\frac{\delta^{(4)}(y^\sigma - x^\sigma(\tau))}{\sqrt{-g}}\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau\,,$$

   where m is the rest mass of the particle. Show that covariant conservation of energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, implies that $x^\mu(\tau)$ satisfies the geodesic equation.

3. Find a coordinate transformation $\tilde{r} = \psi(\tilde{r})^2\tilde{r}$, such that the spatial part of the Schwarzschild metric is in the form

$$ds^2 = \psi^4(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2),$$

   i.e., find $\psi(\tilde{r})$, with the condition that $\tilde{r} \to r$ as $r \to \infty$.

   - What range of Schwarzschild r is covered by the new coordinates?
   - Calculate the area A of a sphere of constant $\tilde{r}$. At what value of $\tilde{r}$ is the area a minimum?

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