General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 09 - Solutions

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Problem 01

Let Ω be a scalar field on the Manifold M and $\widetilde{g} := \Omega \cdot g$ a conformal Transformation of the metric. With the Christoffel-Symbols

$$\Gamma^{\varkappa}_{\mu\nu} = \frac{g^{\varkappa\lambda}}{2} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right)$$

the new Christoffel-Symbols are given by

$$\begin{aligned}
\widetilde{\Gamma}^{\varkappa}_{\mu\nu} &= \frac{\widetilde{g}^{\varkappa\lambda}}{2} \left(\partial_\mu \widetilde{g}_{\lambda\nu} + \partial_\nu \widetilde{g}_{\lambda\mu} - \partial_\lambda \widetilde{g}_{\mu\nu} \right) \\
&= \frac{g^{\varkappa\lambda}}{2\Omega} \cdot \left[(\Omega \partial_\mu g_{\lambda\nu} + g_{\lambda\nu} \partial_\mu \Omega) + (\Omega \partial_\nu g_{\lambda\mu} + g_{\lambda\mu} \partial_\nu \Omega) - (\Omega \partial_\lambda g_{\mu\nu} + g_{\mu\nu} \partial_\lambda \Omega) \right] \\
&= \Gamma^{\varkappa}_{\mu\nu} + \frac{1}{2\Omega} \cdot \left(\delta^{\varkappa}_{\nu} \partial_\mu \Omega + \delta^{\varkappa}_{\mu} \partial_\nu \Omega - g^{\varkappa\lambda} g_{\mu\nu} \partial_\lambda \Omega \right)
\end{aligned}$$

The new Riemann curvature tensor is thus given by

$$\widetilde{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \widetilde{\Gamma}^\rho_{\sigma\nu} - \partial_\nu \widetilde{\Gamma}^\rho_{\sigma\mu} + \widetilde{\Gamma}^\rho_{\mu\varkappa}\widetilde{\Gamma}^\varkappa_{\sigma\nu} - \widetilde{\Gamma}^\rho_{\nu\varkappa}\widetilde{\Gamma}^\varkappa_{\sigma\mu}$$

$$= R^{\rho}{}_{\sigma\mu\nu} + \frac{1}{2\Omega} \cdot \left[\delta^\rho_\nu \partial_{\mu\sigma}\Omega + \cancel{\delta^\rho_\sigma \partial_{\mu\nu}\Omega} - \partial_\mu \left(g^{\rho\lambda} g_{\sigma\nu} \partial_\lambda \Omega\right)\right] - \frac{1}{2\Omega} \cdot \left[\delta^\rho_\mu \partial_{\nu\sigma}\Omega + \cancel{\delta^\rho_\sigma \partial_{\nu\mu}\Omega} - \partial_\nu \left(g^{\rho\lambda} g_{\sigma\mu} \partial_\lambda \Omega\right)\right]$$

$$+ \frac{1}{4\Omega^2} \cdot \left(\delta^\rho_\varkappa \partial_\mu \Omega + \delta^\rho_\mu \partial_\varkappa \Omega - g^{\rho\lambda} g_{\mu\varkappa} \partial_\lambda \Omega\right) \cdot \left(\delta^\varkappa_\nu \partial_\sigma \Omega + \delta^\varkappa_\sigma \partial_\nu \Omega - g^{\varkappa\lambda} g_{\sigma\nu} \partial_\lambda \Omega\right)$$

$$+ \frac{\Gamma^\rho_{\mu\varkappa}}{2\Omega} \cdot \left(\delta^\varkappa_\nu \partial_\sigma \Omega + \delta^\varkappa_\sigma \partial_\nu \Omega - g^{\varkappa\lambda} g_{\sigma\nu} \partial_\lambda \Omega\right) + \frac{\Gamma^\varkappa_{\sigma\nu}}{2\Omega} \left(\delta^\rho_\varkappa \partial_\mu \Omega + \delta^\rho_\mu \partial_\varkappa \Omega - g^{\rho\lambda} g_{\mu\varkappa} \partial_\lambda \Omega\right)$$

$$- \frac{1}{4\Omega^2} \cdot \left(\delta^\rho_\varkappa \partial_\nu \Omega + \delta^\rho_\nu \partial_\varkappa \Omega - g^{\rho\lambda} g_{\nu\varkappa} \partial_\lambda \Omega\right) \cdot \left(\delta^\varkappa_\mu \partial_\sigma \Omega + \delta^\varkappa_\sigma \partial_\mu \Omega - g^{\varkappa\lambda} g_{\sigma\mu} \partial_\lambda \Omega\right)$$

$$- \frac{\Gamma^\rho_{\nu\varkappa}}{2\Omega} \cdot \left(\delta^\varkappa_\mu \partial_\sigma \Omega + \delta^\varkappa_\sigma \partial_\mu \Omega - g^{\varkappa\lambda} g_{\sigma\mu} \partial_\lambda \Omega\right) - \frac{\Gamma^\varkappa_{\sigma\mu}}{2\Omega} \left(\delta^\rho_\varkappa \partial_\nu \Omega + \delta^\rho_\nu \partial_\varkappa \Omega - g^{\rho\lambda} g_{\nu\varkappa} \partial_\lambda \Omega\right)$$

$$= R^{\rho}{}_{\sigma\mu\nu} + \frac{1}{2\Omega} \left[\delta^\rho_\nu \partial_{\mu\sigma}\Omega - \delta^\rho_\mu \partial_{\nu\sigma}\Omega + \partial_\nu \left(g^{\rho\lambda} g_{\sigma\mu} \partial_\lambda \Omega\right) - \partial_\mu \left(g^{\rho\lambda} g_{\sigma\nu} \partial_\lambda \Omega\right)\right]$$

$$+ \frac{1}{4\Omega^2} \left[(\partial_\sigma \Omega) \left(\delta^\rho_\mu (\partial_\nu \Omega) - \delta^\rho_\nu (\partial_\mu \Omega)\right) + g^{\varkappa\lambda} (\partial_\varkappa \Omega)(\partial_\lambda \Omega) \left(\delta^\rho_\nu g_{\mu\sigma} - \delta^\rho_\mu g_{\sigma\nu}\right) + g^{\rho\lambda} (\partial_\lambda \Omega) \left(g_{\nu\sigma} (\partial_\mu \Omega) - g_{\mu\sigma} (\partial_\nu \Omega)\right)\right]$$

$$+ \frac{1}{2\Omega} \Big[g^{\varkappa\lambda} (\partial_\lambda \Omega) \left(\Gamma^\rho_{\nu\varkappa} g_{\sigma\mu} - \Gamma^\rho_{\mu\varkappa} g_{\sigma\nu}\right) + g^{\rho\lambda} (\partial_\lambda \Omega) \underbrace{\left(\Gamma^\varkappa_{\sigma\mu} g_{\nu\varkappa} - \Gamma^\varkappa_{\sigma\nu} g_{\mu\varkappa}\right)}_{\partial_\mu g_{\nu\sigma} - \partial_\nu g_{\mu\sigma}} + (\partial_\varkappa \Omega) \left(\Gamma^\varkappa_{\sigma\nu} \delta^\rho_\mu - \Gamma^\varkappa_{\sigma\mu} \delta^\rho_\nu\right) \Big]$$

$$= R^{\rho}{}_{\sigma\mu\nu} + \frac{1}{2\Omega} \left[\delta^\rho_\nu \partial_{\mu\sigma}\Omega - \delta^\rho_\mu \partial_{\nu\sigma}\Omega + g_{\sigma\mu} \partial_\nu \left(g^{\rho\lambda} \partial_\lambda \Omega\right) - g_{\sigma\nu} \partial_\mu \left(g^{\rho\lambda} \partial_\lambda \Omega\right)\right]$$

$$+ \frac{1}{4\Omega^2} \left[(\partial_\sigma \Omega) \left(\delta^\rho_\mu (\partial_\nu \Omega) - \delta^\rho_\nu (\partial_\mu \Omega)\right) + g^{\varkappa\lambda} (\partial_\varkappa \Omega)(\partial_\lambda \Omega) \left(\delta^\rho_\nu g_{\mu\sigma} - \delta^\rho_\mu g_{\sigma\nu}\right) + g^{\rho\lambda} (\partial_\lambda \Omega) \left(g_{\nu\sigma} (\partial_\mu \Omega) - g_{\mu\sigma} (\partial_\nu \Omega)\right)\right]$$

$$+ \frac{1}{2\Omega} \Big[g^{\varkappa\lambda} (\partial_\lambda \Omega) \left(\Gamma^\rho_{\nu\varkappa} g_{\sigma\mu} - \Gamma^\rho_{\mu\varkappa} g_{\sigma\nu}\right) + (\partial_\varkappa \Omega) \left(\Gamma^\varkappa_{\sigma\nu} \delta^\rho_\mu - \Gamma^\varkappa_{\sigma\mu} \delta^\rho_\nu\right) \Big]$$

Thus follows the Ricci-Tensor

$$\widetilde{R}_{\sigma\nu} = \widetilde{R}^{\mu}{}_{\sigma\mu\nu} = R_{\sigma\nu} + \frac{1}{2\Omega} \Big[(2-n)\partial_{\sigma\nu}\Omega - \underbrace{\partial_\mu \left(g^{\mu\lambda} g_{\sigma\nu} \partial_\lambda \Omega\right)}_{\substack{g^{\mu\lambda}(\partial_\lambda\Omega)\partial_\mu g_{\sigma\nu} \\ + g_{\sigma\nu}\partial_\mu\left(g^{\mu\lambda}\partial_\lambda\Omega\right)}} \Big] + \frac{(n-2)}{4\Omega^2} \left[(\partial_\sigma \Omega)(\partial_\nu \Omega) - g^{\mu\lambda} g_{\sigma\nu} (\partial_\lambda \Omega)(\partial_\mu \Omega)\right]$$

$$+ \frac{1}{2\Omega} \Big[g^{\varkappa\lambda} (\partial_\lambda \Omega) \big[\underbrace{\Gamma^\mu_{\nu\varkappa} g_{\sigma\mu} + \Gamma^\mu_{\sigma\varkappa} g_{\nu\mu}}_{\partial_\varkappa g_{\sigma\nu}} - \Gamma^\mu_{\mu\varkappa} g_{\sigma\nu} \big] + (n-2)(\partial_\varkappa \Omega) \Gamma^\varkappa_{\sigma\nu} \Big]$$

$$= R_{\sigma\nu} + \frac{1}{2\Omega} \Big[(n-2) \left[\Gamma^\varkappa_{\sigma\nu} \partial_\varkappa \Omega - \partial_{\sigma\nu} \Omega\right] - g_{\sigma\nu} \partial_\mu \left(g^{\mu\lambda} \partial_\lambda \Omega\right) - g^{\varkappa\lambda} (\partial_\lambda \Omega) \Gamma^\mu_{\mu\varkappa} g_{\sigma\nu} \Big]$$

$$+ \frac{(n-2)}{4\Omega^2} \left[(\partial_\sigma \Omega)(\partial_\nu \Omega) - g^{\mu\lambda} g_{\sigma\nu} (\partial_\lambda \Omega)(\partial_\mu \Omega)\right]$$

and similarly the Ricci-Scalar

$$\widetilde{R} = \widetilde{g}^{\sigma\nu}\widetilde{R}_{\sigma\nu} = \frac{R}{\Omega} + \frac{1}{2\Omega^2}\Big[(n-2)g^{\nu\sigma}\Gamma^{\varkappa}_{\sigma\nu}\partial_{\varkappa}\Omega - 2(n-1)g^{\nu\sigma}\partial_{\sigma\nu}\Omega - ng^{\varkappa\lambda}(\partial_\lambda\Omega)\Gamma^{\mu}_{\mu\varkappa}\Big] - \frac{(n-1)(n-2)}{4\Omega^3}\cdot g^{\nu\sigma}(\partial_\sigma\Omega)(\partial_\nu\Omega)$$

To show that the Weyl-Tensor C is left invariant, it suffices to show that its components are at any point p left invariant in some arbitrary coordinate system. Choosing Riemann-normal-coordinates (with respect to the original metric g), that is

$$g_{\rho\sigma}\big|_p = \eta_{\rho\sigma} \quad , \quad \partial_\mu g_{\rho\sigma}\big|_p = 0$$

and thus

$$\Gamma^{\mu}_{\rho\sigma}\big|_p = 0 \quad , \quad \partial_\mu(g^{\rho\sigma})\big|_p = 0$$

yields

$$\widetilde{R}^{\rho}{}_{\sigma\mu\nu}\big|_p = R^{\rho}{}_{\sigma\mu\nu} + \frac{1}{2\Omega}\left[\delta^\rho_\nu\partial_{\mu\sigma}\Omega - \delta^\rho_\mu\partial_{\nu\sigma}\Omega + g_{\sigma\mu}g^{\rho\lambda}\partial_{\nu\lambda}\Omega - g_{\sigma\nu}g^{\rho\lambda}\partial_{\mu\lambda}\Omega\right]$$

$$+\frac{1}{4\Omega^2}\left[(\partial_\sigma\Omega)\left(\delta^\rho_\mu(\partial_\nu\Omega) - \delta^\rho_\nu(\partial_\mu\Omega)\right) + g^{\varkappa\lambda}(\partial_\varkappa\Omega)(\partial_\lambda\Omega)\left(\delta^\rho_\nu g_{\mu\sigma} - \delta^\rho_\mu g_{\sigma\nu}\right) + g^{\rho\lambda}(\partial_\lambda\Omega)\left(g_{\nu\sigma}(\partial_\mu\Omega) - g_{\mu\sigma}(\partial_\nu\Omega)\right)\right]$$

$$+\frac{1}{2\Omega}\Big[g^{\varkappa\lambda}(\partial_\lambda\Omega)\left(\Gamma^\rho_{\nu\varkappa}g_{\sigma\mu} - \Gamma^\rho_{\mu\varkappa}g_{\sigma\nu}\right) + (\partial_\varkappa\Omega)\left(\Gamma^\varkappa_{\sigma\nu}\delta^\rho_\mu - \Gamma^\varkappa_{\sigma\mu}\delta^\rho_\nu\right)\Big]$$

for the Riemann-tensor,

$$\widetilde{R}_{\sigma\nu}\big|_p = R_{\sigma\nu} - \frac{1}{2\Omega}\Big[(n-2)\partial_{\sigma\nu}\Omega + g_{\sigma\nu}g^{\mu\lambda}\partial_\mu\left(\partial_\lambda\Omega\right)\Big] + \frac{(n-2)}{4\Omega^2}\left[(\partial_\sigma\Omega)(\partial_\nu\Omega) - g^{\mu\lambda}g_{\sigma\nu}(\partial_\lambda\Omega)(\partial_\mu\Omega)\right]$$

for the Ricci-tensor and

$$\widetilde{R}\big|_p = \frac{R}{\Omega} - \frac{(n-1)}{\Omega^2}g^{\nu\sigma}\partial_{\sigma\nu}\Omega - \frac{(n-1)(n-2)}{4\Omega^3}\cdot g^{\nu\sigma}(\partial_\sigma\Omega)(\partial_\nu\Omega)$$

for the Ricci-scalar. Finally, the Weyl tensor takes the form

$$\widetilde{C}^{\rho}{}_{\sigma\mu\nu}\big|_p = \widetilde{g}^{\rho\lambda}\widetilde{C}_{\lambda\sigma\mu\nu} = \widetilde{R}^{\rho}{}_{\sigma\mu\nu} - \frac{g^{\rho\lambda}}{(n-2)}\Big[g_{\mu\lambda}\widetilde{R}_{\sigma\nu} - g_{\mu\sigma}\widetilde{R}_{\lambda\nu} - g_{\nu\lambda}\widetilde{R}_{\sigma\mu} + g_{\nu\sigma}\widetilde{R}_{\lambda\mu}\Big] + \frac{g^{\rho\lambda}\Omega}{(n-1)(n-2)}\left[g_{\mu\lambda}g_{\sigma\nu} - g_{\mu\sigma}g_{\lambda\nu}\right]\cdot\widetilde{R}$$

$$= C^{\rho}{}_{\sigma\mu\nu} + \frac{1}{2\Omega}\left[\delta^\rho_\nu\partial_{\mu\sigma}\Omega - \delta^\rho_\mu\partial_{\nu\sigma}\Omega + g_{\sigma\mu}g^{\rho\lambda}\partial_{\nu\lambda}\Omega - g_{\sigma\nu}g^{\rho\lambda}\partial_{\mu\lambda}\Omega\right]$$

$$+\frac{1}{4\Omega^2}\Big[(\partial_\sigma\Omega)\left(\delta^\rho_\mu(\partial_\nu\Omega) - \delta^\rho_\nu(\partial_\mu\Omega)\right) + g^{\lambda\varphi}(\partial_\lambda\Omega)(\partial_\varphi\Omega)\left(\delta^\rho_\nu g_{\mu\sigma} - \delta^\rho_\mu g_{\sigma\nu}\right) + g^{\rho\lambda}(\partial_\lambda\Omega)\left(g_{\nu\sigma}(\partial_\mu\Omega) - g_{\mu\sigma}(\partial_\nu\Omega)\right)\Big]$$

$$+\frac{1}{2\Omega(n-2)}\Big[(n-2)\delta^\rho_\mu\partial_{\nu\sigma}\Omega + 2\delta^\rho_\mu g_{\sigma\nu}g^{\varphi\lambda}\partial_{\varphi\lambda}\Omega - (n-2)g_{\mu\sigma}g^{\rho\lambda}\partial_{\lambda\nu}\Omega$$

$$- 2\delta^\rho_\nu g_{\mu\sigma}g^{\varphi\chi}\partial_{\varphi\chi}\Omega - (n-2)\delta^\rho_\nu\partial_{\sigma\mu}\Omega + (n-2)g_{\nu\sigma}g^{\rho\lambda}\partial_{\lambda\mu}\Omega\Big]$$

$$+\frac{1}{4\Omega^2}\Big[-\delta^\rho_\mu(\partial_\sigma\Omega)(\partial_\nu\Omega) + g_{\mu\sigma}g^{\rho\lambda}(\partial_\lambda\Omega)(\partial_\nu\Omega) + \delta^\rho_\nu(\partial_\sigma\Omega)(\partial_\mu\Omega) - g^{\rho\lambda}g_{\nu\sigma}(\partial_\lambda\Omega)(\partial_\mu\Omega)$$

$$+ 2\left(\delta^\rho_\mu g_{\sigma\nu} - \delta^\rho_\nu g_{\mu\sigma}\right)g^{\lambda\varphi}(\partial_\lambda\Omega)(\partial_\varphi\Omega)\Big]$$

$$+\frac{g^{\varphi\lambda}}{\Omega(n-2)}\left(\delta^\rho_\nu g_{\mu\sigma} - \delta^\rho_\mu g_{\nu\sigma}\right)\partial_{\varphi\lambda}\Omega + \frac{g^{\lambda\varphi}}{4\Omega^2}(\partial_\lambda\Omega)(\partial_\varphi\Omega)\left(\delta^\rho_\nu g_{\mu\sigma} - \delta^\rho_\mu g_{\sigma\nu}\right) = C^{\rho}{}_{\sigma\mu\nu}\big|_p$$

□

Problem 02

Through

$$\nabla_\mu T^{\mu\nu}\big|_y = \partial_\mu T^{\mu\nu}\big|_y + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}$$

$$= \int \left[\frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{|g(x(\tau))|}} \cdot \partial_{y^\mu} \delta(y - x(\tau)) + \frac{\delta(y - x(\tau))}{\sqrt{|g(x(\tau))|}} \cdot \left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right) \right] d\tau$$

we see that for arbitrary scalar field F

$$0 = \int_M \overbrace{(\nabla_\mu T^{\mu\nu})}^{0}\big|_y \cdot F(y)\sqrt{|g(y)|}\ dy$$

$$= \int_M \int \left[\frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{|g(x(\tau))|}} \cdot \partial_{y^\mu} \delta(y - x(\tau)) + \frac{\delta(y - x(\tau))}{\sqrt{|g(x(\tau))|}} \cdot \left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right) \right] d\tau\ F(y)\sqrt{|g(y)|}\ dy$$

$$= \int \frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{|g(x(\tau))|}} \underbrace{\int_M F(y)\sqrt{|g(y)|} \cdot \partial_{y^\mu} \delta(y - x(\tau))\ dy}_{\substack{-\int_M \delta(y - x(\tau)) \partial_{y^\mu} \left(F(y)\sqrt{|g(y)|}\right)\ dy \\ \text{due to } (0.3)}}\ d\tau$$

$$+ \int \frac{\left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right)}{\sqrt{|g(x(\tau))|}} \int_M \delta(y - x(\tau)) F(y) \sqrt{|g(y)|}\ dy\ d\tau$$

$$= \int \left[-\frac{\dot{x}^\nu}{\sqrt{|g(x(\tau))|}} \cdot \underbrace{\dot{x}^\mu \partial_\mu \left(F\sqrt{|g|} \right)}_{\frac{d}{d\tau}\left(F(x(\tau))\sqrt{|g(x(\tau))|}\right)} (x(\tau)) + \frac{\left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right)}{\sqrt{|g(x(\tau))|}} F(x(\tau))\sqrt{|g(x(\tau))|} \right] d\tau$$

$$\overset{(0.3)}{=} \int F \left[\sqrt{|g|} \frac{d}{d\tau} \left(\frac{\dot{x}^\nu}{\sqrt{|g|}} \right) + \left(\Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right) \right] d\tau = \int F \left[\ddot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda + \dot{x}^\nu \underbrace{\sqrt{|g|} \frac{d}{d\tau} \left(\frac{1}{\sqrt{|g|}} \right)}_{-\dot{x}^\lambda \frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}}} + \Gamma^\mu_{\mu\lambda} \dot{x}^\lambda \dot{x}^\nu \right] d\tau$$

$$\overset{(0.2)}{=} \int F \left[\ddot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda - \dot{x}^\nu \dot{x}^\lambda \Gamma^\mu_{\mu\lambda} + \dot{x}^\nu \dot{x}^\lambda \Gamma^\mu_{\mu\lambda} \right] d\tau = \int F(x(\tau)) \left[\ddot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda \right] d\tau$$

Since F was arbitrary, we conclude

$$\boxed{\ddot{x}^\nu + \Gamma^\nu_{\mu\lambda} \dot{x}^\mu \dot{x}^\lambda = 0} \tag{0.1}$$

Note that use has been made of the fact

$$\Gamma^\mu_{\mu\lambda} = \frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}} \tag{0.2}$$

and that for n-dimensional distributions Φ by definition

$$\int\limits_{\mathbb{R}^n} F(\mathbf{x})\partial_\mu \Phi(\mathbf{x})\ d^n\mathbf{x} \overset{\text{def}}{=} -\int\limits_{\mathbb{R}^n} \Phi(\mathbf{x})\partial_\mu F(\mathbf{x})\ d^n\mathbf{x} \tag{0.3}$$

□

Problem 03

Motivation for the transformation

Let $r = h(\widetilde{r})$ be a coordinate transformation, such that

$$\left(1 - \frac{2M}{r}\right)^{-1} dr^2 = \frac{h^2}{\widetilde{r}^2}\ d\widetilde{r}^2 \tag{0.4}$$

Then, keeping in mind that

$$dr = \frac{\partial h}{\partial \widetilde{r}}\ d\widetilde{r}$$

yields the differential equation

$$\left(1 - \frac{2M}{h}\right)^{-1} \left(\frac{dh}{d\widetilde{r}}\right)^2 = \frac{h^2}{\widetilde{r}^2}$$

with solution

$$h(\widetilde{r}) = \frac{C^2\widetilde{r}^2 + 2MC\widetilde{r} + M^2}{2\widetilde{r}C} \quad , \quad C \in \mathbb{R}\setminus\{0\}$$

Setting $C = 2$ results in

$$\boxed{r = h(\widetilde{r}) = \underbrace{\left[\frac{2\widetilde{r} + M}{2\widetilde{r}}\right]}_{\psi(\widetilde{r})}^2 \cdot \widetilde{r} =: \psi^2(\widetilde{r})\widetilde{r}} \tag{0.5}$$

Obviously in coordinates $(t, \widetilde{r}, \vartheta, \varphi)$ the spatial part of the Schwarzschild-metric takes the form

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\ d\Omega^2 \overset{(0.4)}{=} \psi^4(\widetilde{r}) \cdot \left(d\widetilde{r}^2 + \widetilde{r}^2\ d\Omega^2\right)$$

Domain and injectivity of transformation

As $r > 0$, also $\widetilde{r} > 0$ must hold. The function h has in $(0, \infty)$ its global minimum at $\widetilde{r}_{\text{min}} = M/2$, where it takes the value $2M$. On the other hand $h(\widetilde{r}) \overset{\widetilde{r}\to\infty}{\longrightarrow} \infty$, so that r takes on (all) values in $[2M, \infty)$. Solving (0.5) for $\widetilde{r}$ leads to

$$\widetilde{r} = \frac{r - M}{2} \pm \frac{r}{2}\sqrt{1 - \frac{2M}{r}}$$

Demanding that $\widetilde{r} \geq M/2$, that is, restricting the domain of $\widetilde{r}$ to $[M/2, \infty)$ and thus getting

$$\widetilde{r}(r) = h^{-1}(r) = \frac{r - M}{2} + \frac{r}{2}\sqrt{1 - \frac{2M}{r}} \tag{0.6}$$

secures that $h : [M/2, \infty) \to [2M, \infty)$ is bijective, thus a valid transformation. Due to

$$\lim_{r\to\infty} \frac{\widetilde{r}(r)}{r} = 1$$

it shows the asymptotic behavior

$$\widetilde{r}(r) \sim r \quad \text{as } r \to \infty$$

Area of a sphere

Let $S := \{\widetilde{r} = \widetilde{r}_0 : \text{const}\}$ be the sub-manifold of constant coordinate $\widetilde{r}$, then the restriction of g on the tangent-space TS takes the form

$$g\big|_{TS} = \psi^4 \widetilde{r}_0^2 \; d\Omega^2$$

with

$$\sqrt{|\det g|} = \psi^4 \widetilde{r}_0^2 \sin\vartheta$$

in standard coordinates (ϑ, φ). Thus integrating the volume-form of S over S results in the area

$$A := \int_S \sqrt{|\det g|} \; d\vartheta \; d\varphi = \int_S \psi^4 \widetilde{r}_0^2 \sin\vartheta \; d\vartheta \; d\varphi = 4\pi\psi^4(\widetilde{r}_0)\widetilde{r}_0^2 = \pi \frac{(2\widetilde{r}_0 + M)^4}{4\widetilde{r}_0^2} = 4\pi r_0^2$$

As mentioned above, r (and thus A) is minimal for $\widetilde{r} = M/2$, corresponding to a minimal area of $A_{\text{min}} = 16\pi M^2$.