General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 08

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Problem 01

Prove the relation

$$2\nabla_{[\mu}\nabla_{\varkappa]}V_\alpha = R^\sigma{}_{\alpha\varkappa\nu}V_\sigma$$

Problem 02

One of the few explicit black-hole solutions to Einstein's equations, the Schwarzschild metric, is spherically symmetric. As a tool to construct the Schwarzschild metric, consider a general spherically symmetric metric,

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

The non-zero Christoffel symbols for this metric are

$$\Gamma^t_{tr} = \partial_r\alpha \qquad \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha \qquad \Gamma^r_{rr} = \partial_r\beta$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r} \qquad \Gamma^r_{\vartheta\vartheta} = -re^{-2\beta} \qquad \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\vartheta \qquad \Gamma^\vartheta_{\phi\phi} = -\sin\vartheta\cos\vartheta \qquad \Gamma^\phi_{\vartheta\phi} = \frac{\cos\vartheta}{\sin\vartheta}$$

The non-zero components of the Riemann tensor are

$$\begin{aligned}
R^t{}_{rtr} &= \partial_r\alpha\ \partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 \\
R^t{}_{\vartheta t\vartheta} &= -re^{-2\beta}\partial_r\alpha \\
R^t{}_{\phi t\phi} &= -re^{-2\beta}\sin^2\vartheta\ \partial_r\alpha \\
R^r{}_{\vartheta r\vartheta} &= re^{-2\beta}\partial_r\beta \\
R^r{}_{\phi r\phi} &= re^{-2\beta}\sin^2\vartheta\ \partial_r\beta \\
R^\vartheta{}_{\phi\vartheta\phi} &= \left(1 - e^{-2\beta}\right)\sin^2\vartheta
\end{aligned}$$

The non-zero components of the Ricci tensor are

$$\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\ \partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\ \partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\vartheta\vartheta} &= e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 \\
R_{\phi\phi} &= R_{\vartheta\vartheta}\sin^2\vartheta
\end{aligned}$$

As an illustration of the calculations involved, verify **four** of the non-zero Christoffel symbols, **three** of the non-zero components of the Riemann tensor, and **three** of the non-zero components of the Ricci tensor. Note that some are easier than others; you can reduce your work by choosing carefully!