General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 08 - Solutions

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Problem 01

Variant 01

$$
\begin{aligned}
2\nabla_{[\nu}\nabla_{\varkappa]}V_\alpha &\overset{\text{def}}{=} \nabla_\nu\nabla_\varkappa V_\alpha - \nabla_\varkappa\nabla_\nu V_\alpha \\
&= \partial_\nu\nabla_\varkappa V_\alpha - \cancel{\Gamma^\lambda_{\nu\varkappa}\nabla_\lambda V_\alpha} - \Gamma^\lambda_{\nu\alpha}\nabla_\varkappa V_\lambda - \partial_\varkappa\nabla_\nu V_\alpha + \cancel{\Gamma^\lambda_{\varkappa\nu}\nabla_\lambda V_\alpha} + \Gamma^\lambda_{\varkappa\alpha}\nabla_\nu V_\lambda \\
&= \partial_\nu\left[\partial_\varkappa V_\alpha - \Gamma^\rho_{\varkappa\alpha}V_\rho\right] - \Gamma^\lambda_{\nu\alpha}\left[\partial_\varkappa V_\lambda - \Gamma^\rho_{\varkappa\lambda}V_\rho\right] - \partial_\varkappa\left[\partial_\nu V_\alpha - \Gamma^\rho_{\nu\alpha}V_\rho\right] + \Gamma^\lambda_{\varkappa\alpha}\left[\partial_\nu V_\lambda - \Gamma^\rho_{\nu\lambda}V_\rho\right] \\
&= -V_\rho\partial_\nu\Gamma^\rho_{\varkappa\alpha} - \cancel{\Gamma^\rho_{\varkappa\alpha}\partial_\nu V_\rho} - \cancel{\Gamma^\lambda_{\nu\alpha}\partial_\varkappa V_\lambda} + \Gamma^\lambda_{\nu\alpha}\Gamma^\rho_{\varkappa\lambda}V_\rho + V_\rho\partial_\varkappa\Gamma^\rho_{\nu\alpha} + \cancel{\Gamma^\rho_{\nu\alpha}\partial_\varkappa V_\rho} + \cancel{\Gamma^\lambda_{\varkappa\alpha}\partial_\nu V_\lambda} - \Gamma^\lambda_{\varkappa\alpha}\Gamma^\rho_{\nu\lambda}V_\rho \\
&= \underbrace{\left[\partial_\varkappa\Gamma^\rho_{\nu\alpha} - \partial_\nu\Gamma^\rho_{\varkappa\alpha} + \Gamma^\lambda_{\nu\alpha}\Gamma^\rho_{\varkappa\lambda} - \Gamma^\lambda_{\varkappa\alpha}\Gamma^\rho_{\nu\lambda}\right]}_{R^\rho{}_{\alpha\varkappa\nu}} \cdot V_\rho = R^\rho{}_{\alpha\varkappa\nu}V_\rho
\end{aligned}
$$

Variant 02 Beginning with the definition

$$R^\rho{}_{\sigma\nu\varkappa}V^\sigma := 2\nabla_{[\nu}\nabla_{\varkappa]}V^\rho$$

we write

$$2\nabla_{[\nu}\nabla_{\varkappa]}V_\alpha = g_{\alpha\rho}2\nabla_{[\nu}\nabla_{\varkappa]}V^\rho = g_{\alpha\rho}R^\rho{}_{\sigma\nu\varkappa} = R_{\alpha\sigma\nu\varkappa}V^\sigma = -R_{\sigma\alpha\nu\varkappa}V^\sigma = -R^\sigma{}_{\alpha\nu\varkappa}V_\sigma = R^\sigma{}_{\alpha\varkappa\nu}V_\sigma$$

□

Problem 02

Using Problem 02 from Set 06 for the diagonal Schwarzschild metric

$$
(g_{\mu\nu}) = \begin{pmatrix} -e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\vartheta \end{pmatrix} \tag{0.1}
$$

we calculate

$$\Gamma^r_{rr} = \frac{1}{2}\frac{\partial_r g_{rr}}{g_{rr}} = \frac{1}{2}\cdot\frac{2e^{2\beta}\partial_r\beta}{e^{2\beta}} = \partial_r\beta$$

$$\Gamma^t_{tr} = \frac{1}{2}\frac{\partial_r g_{tt}}{g_{tt}} = \frac{1}{2}\cdot\frac{e^{2\alpha}(2\partial_r\alpha)}{e^{2\alpha}} = \partial_r\alpha$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{2}\frac{\partial_r g_{\vartheta\vartheta}}{g_{\vartheta\vartheta}} = \frac{1}{2}\cdot\frac{2r}{r^2} = \frac{1}{r}$$

$$\Gamma^\varphi_{r\varphi} = \frac{1}{2}\frac{\partial_r g_{\varphi\varphi}}{g_{\varphi\varphi}} = \frac{1}{2}\cdot\frac{2r\sin^2\vartheta}{r^2\sin^2\vartheta} = \frac{1}{r}$$

Using

$$R^\rho{}_{\alpha\varkappa\nu} := \partial_\varkappa\Gamma^\rho_{\nu\alpha} - \partial_\nu\Gamma^\rho_{\varkappa\alpha} + \Gamma^\lambda_{\nu\alpha}\Gamma^\rho_{\varkappa\lambda} - \Gamma^\lambda_{\varkappa\alpha}\Gamma^\rho_{\nu\lambda}$$

we calculate

$$R^t{}_{\vartheta t\vartheta} = \partial_t\underbrace{\Gamma^t_{\vartheta\vartheta}}_{0} - \partial_\vartheta\underbrace{\Gamma^t_{t\vartheta}}_{0} + \Gamma^t_{\vartheta\vartheta}\underbrace{\Gamma^t_{tt}}_{0} + \underbrace{\Gamma^r_{\vartheta\vartheta}\Gamma^t_{tr}}_{-re^{-2\beta}\partial_r\alpha} + \underbrace{\Gamma^\vartheta_{\vartheta\vartheta}}_{0}\Gamma^t_{t\vartheta} + \Gamma^\varphi_{\vartheta\vartheta}\underbrace{\Gamma^t_{t\varphi}}_{0} - \underbrace{\Gamma^t_{t\vartheta}}_{0}\Gamma^t_{\vartheta t} - \Gamma^r_{t\vartheta}\underbrace{\Gamma^t_{\vartheta r}}_{0} - \Gamma^\vartheta_{t\vartheta}\underbrace{\Gamma^t_{\vartheta\vartheta}}_{0} - \Gamma^\varphi_{t\vartheta}\underbrace{\Gamma^t_{\vartheta\varphi}}_{0}$$

$$= -re^{-2\beta}\partial_r\alpha$$

$$R^r{}_{\vartheta r\vartheta} = \partial_r\underbrace{\Gamma^r_{\vartheta\vartheta}}_{-re^{-2\beta}} - \partial_\vartheta\underbrace{\Gamma^r_{r\vartheta}}_{0} + \Gamma^t_{\vartheta\vartheta}\underbrace{\Gamma^r_{rt}}_{0} + \underbrace{\Gamma^r_{\vartheta\vartheta}\Gamma^r_{rr}}_{-re^{-2\beta}\partial_r\beta} + \underbrace{\Gamma^\vartheta_{\vartheta\vartheta}}_{0}\Gamma^r_{r\vartheta} + \Gamma^\varphi_{\vartheta\vartheta}\underbrace{\Gamma^r_{r\varphi}}_{0} - \underbrace{\Gamma^t_{r\vartheta}}_{0}\Gamma^r_{\vartheta t} - \Gamma^r_{r\vartheta}\underbrace{\Gamma^r_{\vartheta r}}_{0} - \underbrace{\Gamma^\vartheta_{r\vartheta}\Gamma^r_{\vartheta\vartheta}}_{\frac{1}{r}\cdot(-r)e^{-2\beta}} - \Gamma^\varphi_{r\vartheta}\underbrace{\Gamma^r_{\vartheta\varphi}}_{0}$$

$$= -e^{-2\beta} + 2re^{-2\beta}\partial_r\beta - re^{-2\beta}\partial_r\beta + e^{-2\beta} = re^{-2\beta}\partial_r\beta$$

$$R^t{}_{\varphi t\varphi} = \partial_t\underbrace{\Gamma^t_{\varphi\varphi}}_{0} - \partial_\varphi\underbrace{\Gamma^t_{t\varphi}}_{0} + \underbrace{\Gamma^t_{\varphi\varphi}}_{0}\Gamma^t_{tt} + \underbrace{\Gamma^r_{\varphi\varphi}\Gamma^t_{tr}}_{-r(\partial_r\alpha)e^{-2\beta}\sin^2\vartheta} + \Gamma^\vartheta_{\varphi\varphi}\underbrace{\Gamma^t_{t\vartheta}}_{0} + \underbrace{\Gamma^\varphi_{\varphi\varphi}}_{0}\Gamma^t_{t\varphi} - \underbrace{\Gamma^t_{t\varphi}}_{0}\Gamma^t_{\varphi t} - \underbrace{\Gamma^r_{t\varphi}}_{0}\Gamma^t_{\varphi r} - \underbrace{\Gamma^\vartheta_{t\varphi}}_{0}\Gamma^t_{\varphi\vartheta} - \Gamma^\varphi_{t\varphi}\underbrace{\Gamma^t_{\varphi\varphi}}_{0}$$

$$= -re^{-2\beta}\sin^2\vartheta\ \partial_r\alpha$$

Finally, using

$$R_{\mu\nu} := R^\rho{}_{\mu\rho\nu}$$

we calculate

$$R_{rr} = R^t{}_{rtr} + \underbrace{R^r{}_{rrr}}_{0} + \underbrace{R^\vartheta{}_{r\vartheta r}}_{R^r{}_{\vartheta r\vartheta}\cdot\frac{g^{\vartheta\vartheta}}{g^{rr}}} + \underbrace{R^\varphi{}_{r\varphi r}}_{R^r{}_{\varphi r\varphi}\cdot\frac{g^{\varphi\varphi}}{g^{rr}}}$$

$$= \partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 + re^{-2\beta}\partial_r\beta\cdot\frac{e^{2\beta}}{r^2} + re^{-2\beta}\sin^2\vartheta\ \partial_r\beta\cdot\frac{e^{2\beta}}{r^2\sin^2\vartheta}$$

$$= \partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 + \frac{2}{r}\partial_r\beta$$

$$R_{\vartheta\vartheta} = R^t{}_{\vartheta t\vartheta} + R^r{}_{\vartheta r\vartheta} + \underbrace{R^\vartheta{}_{\vartheta\vartheta\vartheta}}_{0} + \underbrace{R^\varphi{}_{\vartheta\varphi\vartheta}}_{R^\vartheta{}_{\varphi\vartheta\varphi}\cdot\frac{g^{\varphi\varphi}}{g^{\vartheta\vartheta}}}$$

$$= -re^{-2\beta}\partial_r\alpha + re^{-2\beta}\partial_r\beta + \left(1-e^{-2\beta}\right)\sin^2\vartheta\cdot\frac{r^2}{r^2\sin^2\vartheta}$$

$$= e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1$$

$$R_{\varphi\varphi} = R^t{}_{\varphi t\varphi} + R^r{}_{\varphi r\varphi} + R^\vartheta{}_{\varphi\vartheta\varphi} + R^\varphi{}_{\varphi\varphi\varphi}$$

$$= -re^{-2\beta}\sin^2\vartheta\ \partial_r\alpha + re^{-2\beta}\sin^2\vartheta\ \partial_r\beta + \left(1-e^{-2\beta}\right)\sin^2\vartheta$$

$$= \left\{e^{-2\beta}\cdot\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1\right\}\cdot\sin^2\vartheta = R_{\vartheta\vartheta}\cdot\sin^2\vartheta$$

Note that use has been made of

$$\underbrace{R^\mu{}_{\nu\mu\nu}}_{\text{no summation}} = \underbrace{g^{\mu\lambda}R_{\lambda\nu\mu\nu}}_{\text{summation over }\lambda} \overset{g \text{ diagonal}}{=} \underbrace{g^{\mu\mu}R_{\mu\nu\mu\nu}}_{\text{no summation}} = \underbrace{g^{\nu\nu}R_{\nu\mu\nu\mu}\cdot\frac{g^{\mu\mu}}{g^{\nu\nu}}}_{\text{no summation}} \overset{g \text{ diagonal}}{=} \underbrace{g^{\nu\lambda}R_{\lambda\mu\nu\mu}\cdot\frac{g^{\mu\mu}}{g^{\nu\nu}}}_{\text{summation over }\lambda} = \underbrace{R^\nu{}_{\mu\nu\mu}\cdot\frac{g^{\mu\mu}}{g^{\nu\nu}}}_{\text{no summation}}$$