GR: Problem Set #7

- A geodesic curve is a generalization of the idea of a straight line to the case of curvilinear coordinates or curved manifolds. By definition, a geodesic curve is a curve that parallel transports its own tangent vector, that is, a curve whose tangent vector satisfies

$$U^\alpha \nabla_\alpha U^\beta = 0\,.$$

- A Killing vector field is a vector field on a manifold that preserves the metric. This field satisfies the equation

$$\nabla_{(\mu} K_{\nu)} = 0\,.$$

- The components of the Riemann curvature tensor, in terms of the Christoffel symbols, are given by

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\rho\mu}\,\Gamma^\rho_{\beta\nu} - \Gamma^\alpha_{\rho\nu}\,\Gamma^\rho_{\beta\mu}\,.$$

1. (Carroll, Problem 3.5.) Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

   (a) Show that lines of constant longitude ($\phi =$ constant) are geodesics, and that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).

   (b) Take a vector with components $V^\mu = (1, 0)$ and parallel-transport it once around a circle of constant latitude. What are the components of the resulting vector, as a function of θ?

2. (Carroll, Problem 3.6.) A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

   where

$$\Phi = -\frac{GM}{r}$$

   may be thought of as the familiar Newtonian gravitational potential. Hence G is Newton's constant and M is the mass of the Earth. For this problem Φ may be assumed to be small.

   (a) Imagine a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock runs faster?

   (b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

3. If $\mathbf{K}$ is a Killing vector field and $\mathbf{p}$ is the tangent vector to a geodesic, show that $\mathbf{K} \cdot \mathbf{p}$ is constant along the geodesic.

4. If **K** is a Killing vector, prove that

$$\nabla_\alpha \nabla_\beta K^\mu = R^\mu{}_{\beta\alpha\nu}\, K^\nu\,.$$

Milton Ruiz, milton.ruiz@uni-jena.de