General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 07 - Solutions

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Problem 01 (Carroll, Problem 3.5)

Preconsiderations

Let M be the considered manifold with coordinate vector-basis $\{\partial_\mu\}$. A curve $x(t) : I \subseteq \mathbb{R} \to M$ is geodesic if

$$\nabla_{\dot{x}}\dot{x} = 0$$

where

$$\dot{x} := \frac{dx^\mu}{dt}\partial_\mu = \dot{x}^\mu \partial_\mu$$

is the tangent vector generated by the curve. Because of

$$\nabla_{\dot{x}}\dot{x} = \dot{x}^\mu \nabla_{\partial_\mu}(\dot{x}^\nu \partial_\nu) = \underbrace{\dot{x}^\mu(\partial_\mu \dot{x}^\nu)}_{\ddot{x}^\nu}\partial_\nu + \dot{x}^\mu \dot{x}^\lambda \nabla_{\partial_\mu}\partial_\lambda = \ddot{x}^\nu \partial_\nu + \dot{x}^\mu \dot{x}^\lambda \Gamma^\nu_{\mu\lambda}\partial_\nu$$

this is equivalent to

$$\ddot{x}^\varkappa + \Gamma^\varkappa_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \tag{0.1}$$

A point-set $P \subseteq M$ is said to be geodesic, if it allows a smooth, geodesic parametrization $x(t)$. It turns out, that if P is indeed geodesic, any parametrization $x(t)$ with $g(\dot{x}, \dot{x}) = \text{const} \neq 0$ is geodesic.

The parallel transport of a vector $V \in T_pM$ along a curve $x : I \subseteq \mathbb{R} \to M$ is defined as the family of tangential vectors $(V(t))_{t\in I}, V(t) \in T_{x(t)}M$ for which

$$\nabla_{\dot{x}(t)}V(t) = 0$$

that is

$$0 = \dot{x}^\mu \nabla_{\partial_\mu} V^\nu \partial_\nu = \underbrace{\dot{x}^\mu(\partial_\mu V^\nu)}_{\dot{V}^\nu}\partial_\nu + \dot{x}^\mu V^\lambda \nabla_{\partial_\mu}\partial_\lambda = \dot{V}^\nu \partial_\nu + \Gamma^\nu_{\mu\lambda}\dot{x}^\mu V^\lambda \partial_\nu$$

or

$$\dot{V}^\varkappa + \Gamma^\varkappa_{\mu\nu}\dot{x}^\mu V^\nu = 0 \tag{0.2}$$

The Christoffel-Symbols are for this special metric given by

$$\Gamma^\varphi_{\vartheta\vartheta} = 0 \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta$$
$$\Gamma^\vartheta_{\vartheta\varphi} = \Gamma^\vartheta_{\varphi\vartheta} = 0 \qquad \Gamma^\varphi_{\varphi\vartheta} = \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta$$
$$\Gamma^\vartheta_{\vartheta\vartheta} = 0 \qquad \Gamma^\varphi_{\varphi\varphi} = 0$$

(compare to problem set 06, Problem 02).

(a) Consider the curve $x(t) : I \subseteq \mathbb{R} \to S^2$ of constant φ, that is

$$x(t) := (\vartheta(t), \varphi(t)) := (\vartheta_0 + v_\vartheta \cdot t, \varphi_0) \quad , \quad \vartheta_0, v_\vartheta, \varphi_0 : \text{const}$$

Then through

$$\ddot{\vartheta} + \Gamma^{\vartheta}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \underbrace{\ddot{\vartheta}}_{0} + \Gamma^{\vartheta}_{\varphi\varphi}\dot{\varphi}\underbrace{\dot{\varphi}}_{0} = 0$$

$$\ddot{\varphi} + \Gamma^{\varphi}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \underbrace{\ddot{\varphi}}_{0} + 2\Gamma^{\varphi}_{\vartheta\varphi}\dot{\vartheta}\underbrace{\dot{\varphi}}_{0} = 0$$

one sees that $x(t)$ is indeed geodesic.
Let w.l.o.g. $\vartheta_0 \notin \{0, \pi\}$. For the curve

$$x(t) := (\vartheta(t), \varphi(t)) := (\vartheta_0,\ \varphi_0 + v_\varphi \cdot t) \quad , \quad \vartheta_0, \varphi_0, v_\varphi : \text{const},\ v_\varphi \neq 0$$

$$g(\dot{x}, \dot{x}) = g_{\varphi\varphi} v_\varphi^2 = \sin^2\vartheta_0 \cdot v_\varphi^2 : \text{const} \neq 0$$

it holds

$$\ddot{\vartheta} + \Gamma^{\vartheta}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \underbrace{\ddot{\vartheta}}_{0} + \Gamma^{\vartheta}_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = -\sin\vartheta\cos\vartheta \cdot v_\varphi^2$$

$$\ddot{\varphi} + \Gamma^{\varphi}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \underbrace{\ddot{\varphi}}_{0} + 2\Gamma^{\varphi}_{\vartheta\varphi}\underbrace{\dot{\vartheta}}_{0}\dot{\varphi} = 0$$

from which follows that, $x(t)$ is geodesic if and only if $\sin\vartheta\cos\vartheta = 0$, that is $\cos\vartheta = 0$ and thus $\vartheta = \pi/2$.

(b) Consider the curve (circle) of constant latitude

$$x(t) := (\vartheta(t), \varphi(t)) := (\vartheta_0, t) \quad , \quad t \in [0, 2\pi]$$

and the vector $V = (1, 0)$ at $(\vartheta_0, \varphi = 0)$. Then, differential equation (0.2) takes the form

$$0 = \dot{V}^\vartheta + \Gamma^{\vartheta}_{\mu\nu}\dot{x}^\mu V^\nu = \dot{V}^\vartheta + \Gamma^{\vartheta}_{\varphi\varphi}\underbrace{\dot{\varphi}}_{1} V^\varphi = \dot{V}^\vartheta - V^\varphi \sin\vartheta_0\cos\vartheta_0$$

$$0 = \dot{V}^\varphi + \Gamma^{\varphi}_{\mu\nu}\dot{x}^\mu V^\nu = \dot{V}^\varphi + \Gamma^{\varphi}_{\varphi\vartheta}\underbrace{\dot{\varphi}}_{1} V^\vartheta + \Gamma^{\varphi}_{\vartheta\varphi}\underbrace{\dot{\vartheta}}_{0} V^\varphi = \dot{V}^\varphi + V^\vartheta\cot\vartheta_0$$

Differentiating the first equation we obtain the initial-value-problem (IVP)

$$\ddot{V}^\vartheta = -V^\vartheta\cos^2\vartheta_0$$

whereas

$$V^\vartheta(0) = 1 \quad , \quad \dot{V}^\vartheta(0) = V^\varphi(0)\sin\vartheta_0\cos\vartheta_0 = 0$$

and thus

$$V^\vartheta(t) = \cos\left[t \cdot \cos\vartheta_0\right] \tag{0.3}$$

Using (0.3) we obtain the IVP

$$\dot{V}^\varphi = -V^\vartheta\cot\vartheta_0 = -\cos\left[t\cdot\cos\vartheta_0\right]\cdot\cot\vartheta_0 \quad , \quad V^\varphi(0) = 0$$

with solution

$$V^\varphi(t) = -\frac{\sin\left[t\cdot\cos\vartheta_0\right]}{\sin\vartheta_0} \tag{0.4}$$

Thus the transport of $V=(1,0)$ along $x(t)$ is given by

$$V(t)=(V^{\vartheta}(t),V^{\varphi}(t))=\left[\cos\left[t\cdot\cos\vartheta_0\right],\ -\frac{\sin\left[t\cdot\cos\vartheta_0\right]}{\sin\vartheta_0}\right] \tag{0.5}$$

The transport along one circle ($t=2\pi$) results in the vector

$$\boxed{V_{\text{circle}}=(V^{\vartheta}_{\text{circle}},V^{\varphi}_{\text{circle}})=\left[\cos\left[2\pi\cos\vartheta_0\right],\ -\frac{\sin\left[2\pi\cos\vartheta_0\right]}{\sin\vartheta_0}\right]} \tag{0.6}$$

Problem 02 (Carroll, Problem 3.6)

(a) We shall parametrize the space-time-paths of the two clocks by their respective proper-times τ_1,τ_2

$$x_i(t):=(t_i,r_i,\vartheta_i,\varphi_i)(\tau_i):=(t_i(\tau_i),R_i,\vartheta_i,\varphi_i)\quad,\quad \vartheta_i,\varphi_i,R_i:\text{const},\ i=1,2$$

(w.l.o.g. $\dot t_i\ge 0$) so that

$$1=|g(\dot x_i,\dot x_i)|=\left|g_{tt}\dot t_i\right|=|(1+2\Phi(R_i))|\cdot\dot t_i$$

and thus

$$\frac{d\tau_i}{dt}=\frac{d\tau_i}{dt_i}=\left(\dot t_i\right)^{-1}=|1+2\Phi(R_i)|\overset{|\Phi|\le 1/2}{=}1-\frac{2GM}{R_i}$$

Hence, the time elapsed τ on each clock as a function of the coordinate time t is given by

$$\boxed{\tau_i(t)=\left(1-\frac{2GM}{R_i}\right)\cdot t} \tag{0.7}$$

In particular, clock 2 (further away from earth's center) runs faster.

(b) Let

$$x(\tau):=(t(\tau),r(\tau),\vartheta(\tau),\varphi(\tau)):=(t(\tau),r_0,\vartheta_0,\varphi(\tau))\quad,\quad r_0,\vartheta_0:=\frac{\pi}{2}:\text{const} \tag{0.8}$$

be the sought geodetic, whereas w.l.o.g. $g(\dot x,\dot x)=-1$ (parametrized by proper-time). Then $x(\tau)$ solves the Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial L}{\partial\dot x^{\mu}}-\frac{\partial L}{\partial x^{\mu}}=0$$

for the Lagrangian

$$L(x,\dot x):=g(\dot x,\dot x)=g_{tt}\dot t^2+g_{rr}\dot r^2+g_{\vartheta\vartheta}\dot\vartheta^2+g_{\varphi\varphi}\dot\varphi^2$$

Specifically,

$$0=\frac{d}{d\tau}\frac{\partial L}{\partial\dot r}-\frac{\partial L}{\partial r}=\underbrace{\frac{d}{d\tau}(2\dot r g_{rr})}_{2\ddot r g_{rr}+2\dot r\partial_\tau g_{rr}=0}-\dot t^2\partial_r g_{tt}-\underbrace{\dot r^2}_{0}\partial_r g_{rr}-\underbrace{\dot\vartheta^2}_{0}\partial_r g_{\vartheta\vartheta}-\dot\varphi^2\partial_r g_{\varphi\varphi}$$

$$=\dot t^2\cdot\frac{2GM}{r_0^2}-\dot\varphi^2\cdot 2r_0\underbrace{\sin^2\vartheta_0}_{1}$$

is equivalent to

$$\left(\frac{d\varphi}{dt}\right)^2=\frac{\dot\varphi^2}{\dot t^2}=\frac{GM}{r_0^3}$$

(w.l.o.g. $\frac{d\varphi}{dt}\ge 0$) or equivalently

$$\boxed{\frac{d\varphi}{dt}=\sqrt{\frac{GM}{r_0^3}}} \tag{0.9}$$

Moreover,

$$0 = \frac{d}{d\tau}\frac{\partial L}{\partial \dot t} - \frac{\partial L}{\partial t} = \frac{d}{d\tau}(2\dot t g_{tt}) = 2\ddot t \underbrace{g_{tt}}_{\text{const}}$$

is equivalent to $\ddot t = 0$, that is,

$$\boxed{t(\tau) = t_0 + v_t \cdot \tau \quad , \quad t_0, v_t : \text{const}} \tag{0.10}$$

Similarly,

$$0 = \frac{d}{d\tau}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = \frac{d}{d\tau}(2\dot\varphi g_{\varphi\varphi}) = 2\ddot\varphi g_{\varphi\varphi}$$

is equivalent to $\ddot\varphi = 0$, that is,

$$\boxed{\varphi(\tau) = \varphi_0 + v_\varphi \cdot \tau \quad , \quad t_0 : \text{const}, \ v_\varphi \overset{(0.9)}{=} v_t \cdot \sqrt{\frac{GM}{r_0^3}}} \tag{0.11}$$

Finally, assumption $g(\dot x, \dot x) = -1$, that is

$$-1 = g_{tt}\dot t^2 + g_{\varphi\varphi}\dot\varphi^2 = \left(\frac{2GM}{r_0} - 1\right) \cdot v_t^2 + r_0^2 \underbrace{\sin^2\vartheta_0}_{1} \cdot v_\varphi^2$$

leads to

$$\boxed{v_t = \sqrt{\frac{1}{1 - \frac{3GM}{r_0}}}} \tag{0.12}$$

Concluding, the geodesic at radius r_0 is given by

$$\boxed{(t(\tau), r(\tau), \vartheta(\tau), \varphi(\tau)) = \left[t_0 + \sqrt{\frac{1}{1 - \frac{3GM}{r_0}}} \cdot \tau, \ r_0, \ \frac{\pi}{2}, \ \varphi_0 + \frac{1}{r_0}\sqrt{\frac{GM}{r_0 - 3GM}} \cdot \tau\right]} \tag{0.13}$$

Note: Because of

$$0 = \frac{d}{d\tau}\frac{\partial L}{\partial \dot\vartheta} - \frac{\partial L}{\partial \vartheta} = \frac{d}{d\tau}(2\dot\vartheta g_{\vartheta\vartheta}) - \dot\varphi^2 \partial_\vartheta g_{\varphi\varphi} = 2\underbrace{\ddot\vartheta}_{0} g_{\vartheta\vartheta} + 2\underbrace{\dot\vartheta}_{0}\frac{dg_{\vartheta\vartheta}}{d\tau} - \dot\varphi^2 \cdot 2r_0^2 \sin\vartheta_0 \underbrace{\cos\vartheta_0}_{0} = 0$$

curve (0.13) is indeed geodesic.

Problem 03

Help-statement

A vector-field $\mathbf{K}$ is Killing if and only if it satisfies

$$g(\nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}) + g(\nabla_{\mathbf{Y}}\mathbf{K}, \mathbf{X}) = 0 \tag{0.14}$$

for any vector fields $\mathbf{X} = X^\mu \partial_\mu, \mathbf{Y} = Y^\mu \partial_\mu$.

Proof: Condition $\nabla_{(\nu}K_{\mu)} = 0$, that is,

$$0 = \underbrace{(\nabla_{\partial_\nu} g(\mathbf{K}, \cdot))_\mu}_{\substack{\nabla_{\partial_\nu}(g(\mathbf{K},\partial_\mu)) \\ -g(\mathbf{K},\nabla_{\partial_\nu}\partial_\mu)}} + \underbrace{\left(\nabla_{\partial_\mu} g(\mathbf{K}, \cdot)\right)_\nu}_{\substack{\nabla_{\partial_\mu}(g(\mathbf{K},\partial_\nu)) \\ -g(\mathbf{K},\nabla_{\partial_\mu}\partial_\nu)}}$$

$$= \left[g(\mathbf{K}, \nabla_{\partial_\nu}\partial_\mu) + g(\nabla_{\partial_\nu}\mathbf{K}, \partial_\mu)\right] - g(\mathbf{K}, \nabla_{\partial_\nu}\partial_\mu) + \left[g(\nabla_{\partial_\mu}\mathbf{K}, \partial_\nu) + g(\mathbf{K}, \nabla_{\partial_\mu}\partial_\nu)\right] - g(\mathbf{K}, \nabla_{\partial_\mu}\partial_\nu)$$

$$= g(\nabla_{\partial_\nu}\mathbf{K}, \partial_\mu) + g(\nabla_{\partial_\mu}\mathbf{K}, \partial_\nu)$$

is equivalent to

$$0 = g\left(\nabla_{\partial_\nu}\mathbf{K}, \partial_\mu\right) X^\nu Y^\mu + g\left(\nabla_{\partial_\mu}\mathbf{K}, \partial_\nu\right) X^\nu Y^\mu$$

$$= g\left(\nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}\right) + g\left(\nabla_{\mathbf{Y}}\mathbf{K}, \mathbf{X}\right) \quad \forall \text{ vectorfields } \mathbf{X}, \mathbf{Y}$$

Note that use has been made of the metric-compatibility of the Levi-Civita-connection

$$\nabla_{\mathbf{X}}(g(\mathbf{Y}, \mathbf{Z})) = g\left(\nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\right) + g\left(\mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}\right) \quad \forall \ \mathbf{X}, \mathbf{Y}, \mathbf{Z}$$

□

Proof of main statement

To be shown is $\mathbf{p}g(\mathbf{K}, \mathbf{p}) = 0$. But indeed, it follows from metric-compatibility of the Levi-Civita-connection

$$\nabla_{\mathbf{X}}(g(\mathbf{Y}, \mathbf{Z})) = g\left(\nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\right) + g\left(\mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}\right) \quad \forall \ \mathbf{X}, \mathbf{Y}, \mathbf{Z}$$

that

$$\underbrace{\mathbf{p}g(\mathbf{K}, \mathbf{p})}_{\nabla_{\mathbf{p}}g(\mathbf{K},\mathbf{p})} = \underbrace{g\left(\nabla_{\mathbf{p}}\mathbf{K}, \mathbf{p}\right)}_{0} + g\big(\mathbf{K}, \underbrace{\nabla_{\mathbf{p}}\mathbf{p}}_{0}\big) = 0$$

Note that use has been made of the help-statement to show

$$0 \overset{(0.14)}{=} g(\nabla_{\mathbf{p}}\mathbf{K}, \mathbf{p}) + g(\nabla_{\mathbf{p}}\mathbf{K}, \mathbf{p}) = 2g(\nabla_{\mathbf{p}}\mathbf{K}, \mathbf{p})$$

□

Problem 04

Preconsiderations

As the equation to be shown

$$\nabla_\alpha \nabla_\beta K^\mu = R^\mu_{\beta\alpha\nu} K^\nu$$

is a tensorial one, it suffices to show it's validity in any arbitrary coordinates. As is known, on any given point $p \in M$, there exists a local chart such that

$$\left(g_{\mu\nu}\right)\Big|_p = \mathrm{diag}(\epsilon_1, .., \epsilon_n), \quad \epsilon_i \in \{\pm 1\}$$

and $\partial_\alpha g_{\mu\nu}\ |_p = 0$ (Riemann-normal coordinates).

As $\mathbf{K}$ is Killing, that is

$$0 = \partial_\mu K_\nu + \partial_\nu K_\mu - 2\Gamma^\lambda_{\nu\mu} K_\lambda = \partial_\mu (K^\varkappa g_{\varkappa\nu}) + \partial_\nu (K^\varkappa g_{\varkappa\mu}) - 2\Gamma^\lambda_{\nu\mu} g_{\lambda\varkappa} K^\varkappa$$

$$= K^\varkappa \underbrace{\left[\partial_\mu g_{\varkappa\nu} + \partial_\nu g_{\varkappa\mu}\right]}_{2\Gamma^\lambda_{\mu\nu} g_{\lambda\varkappa} + \partial_\varkappa g_{\mu\nu}} + g_{\varkappa\nu}\partial_\mu K^\varkappa + g_{\varkappa\mu}\partial_\nu K^\varkappa - 2\Gamma^\lambda_{\nu\mu} g_{\lambda\varkappa} K^\varkappa$$

$$= K^\varkappa \partial_\varkappa g_{\mu\nu} + g_{\varkappa\nu}\partial_\mu K^\varkappa + g_{\varkappa\mu}\partial_\nu K^\varkappa$$

it also fulfills

$$0 = \partial_\sigma \left(K^\varkappa \partial_\varkappa g_{\mu\nu} + g_{\varkappa\nu}\partial_\mu K^\varkappa + g_{\varkappa\mu}\partial_\nu K^\varkappa\right) \overset{\substack{\text{normal}\\ \text{coordinates}}}{=} K^\varkappa \partial_{\sigma\varkappa} g_{\mu\nu} + g_{\varkappa\nu}\partial_{\sigma\mu} K^\varkappa + g_{\varkappa\mu}\partial_{\sigma\nu} K^\varkappa \qquad (0.15)$$

With

$$\nabla_\alpha \nabla_\beta K^\mu = \partial_\alpha(\nabla_\beta K^\mu) + \Gamma^\mu_{\alpha\lambda}\nabla_\beta K^\lambda - \Gamma^\lambda_{\alpha\beta}\nabla_\lambda K^\mu$$

$$= \partial_\alpha\left(\partial_\beta K^\mu + \Gamma^\mu_{\beta\varkappa}K^\varkappa\right) + \Gamma^\mu_{\alpha\lambda}\left(\partial_\beta K^\lambda + \Gamma^\lambda_{\beta\varkappa}K^\varkappa\right) - \Gamma^\lambda_{\alpha\beta}\left(\partial_\lambda K^\mu + \Gamma^\mu_{\lambda\varkappa}K^\varkappa\right)$$

$$= \partial_{\alpha\beta}K^\mu + (\partial_\alpha \Gamma^\mu_{\beta\varkappa})K^\varkappa + \Gamma^\mu_{\beta\varkappa}\partial_\alpha K^\varkappa + \Gamma^\mu_{\alpha\varkappa}\partial_\beta K^\varkappa + \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\beta\varkappa}K^\varkappa - \Gamma^\lambda_{\alpha\beta}\partial_\lambda K^\mu - \Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\lambda\varkappa}K^\varkappa$$

$$\overset{\substack{\text{normal}\\ \text{coordinates}}}{=} \partial_{\alpha\beta}K^\mu + (\partial_\alpha \Gamma^\mu_{\beta\varkappa})K^\varkappa$$

and

$$R^\mu{}_{\beta\alpha\varkappa}K^\varkappa \overset{\substack{\text{normal}\\ \text{coordinates}}}{=} (\partial_\alpha \Gamma^\mu_{\beta\varkappa})K^\varkappa - (\partial_\varkappa \Gamma^\mu_{\beta\alpha})K^\varkappa$$

it suffices to show

$$\partial_{\alpha\beta}K^\mu = -(\partial_\varkappa \Gamma^\mu_{\beta\alpha})K^\varkappa \tag{0.16}$$

Final proof

Indeed,

$$-(\partial_\varkappa \Gamma^\mu_{\beta\alpha})K^\varkappa \overset{\substack{\text{normal}\\ \text{coordinates}}}{=} -\frac{g^{\mu\lambda}}{2}\left(\partial_{\varkappa\beta}g_{\alpha\lambda} + \partial_{\varkappa\alpha}g_{\beta\lambda} - \partial_{\varkappa\lambda}g_{\beta\alpha}\right)K^\varkappa$$

$$\overset{(0.15)}{=} -\frac{g^{\mu\lambda}}{2}\left(-g_{\varkappa\lambda}\partial_{\beta\alpha}K^\varkappa - g_{\varkappa\alpha}\partial_{\beta\lambda}K^\varkappa - g_{\varkappa\lambda}\partial_{\alpha\beta}K^\varkappa - g_{\varkappa\beta}\partial_{\alpha\lambda}K^\varkappa + g_{\varkappa\alpha}\partial_{\lambda\beta}K^\varkappa + g_{\varkappa\beta}\partial_{\lambda\alpha}K^\varkappa\right)$$

$$= g^{\mu\lambda}g_{\varkappa\lambda}\partial_{\alpha\beta}K^\varkappa = \partial_{\alpha\beta}K^\mu$$

□