General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 06

December 15, 2009

Problem 01

Show that a conformal transformation of a metric, i.e. $g_{\mu\nu} = \Omega(x^\lambda)\widetilde{g}_{\mu\nu}$ for an arbitrary function Ω, preserves all angles. (First work out how to define angles; think of how an angle between to vectors is defined.) Show that all null curves ($g_{\mu\nu}X^\mu X^\nu = 0$) remain null curves.

Problem 02 (Carroll, Problem 3.3)

Imagine that we have a *diagonal* metric $g_{\mu\nu}$. Show that the Christoffel symbols are given by

$$\Gamma^\lambda_{\mu\nu} = 0 \tag{0.1}$$

$$\Gamma^\lambda_{\mu\mu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu} \tag{0.2}$$

$$\Gamma^\lambda_{\mu\lambda} = \partial_\mu \left(\ln\sqrt{|g_{\lambda\lambda}|}\right) \tag{0.3}$$

$$\Gamma^\lambda_{\lambda\lambda} = \partial_\lambda \left(\ln\sqrt{|g_{\lambda\lambda}|}\right) \tag{0.4}$$

In these expressions, $\mu \neq \nu \neq \lambda$, and repeated indices are *not* summed over.

Problem 03

Calculate the Christoffel symbols for the metric of spherical polar coordinates

$$ds^2 = dr^2 + r^2\ d\theta^2 + r^2 \sin^2\theta\ d\phi^2$$

Problem 04 (Carroll, Problem 3.2)

You are familiar with the operations of gradient ($\nabla\phi$), divergence ($\nabla \cdot \mathbf{V}$) and curl ($\nabla \times \mathbf{V}$) in ordinary vector analysis in three-dimensional Euclidean space. Using covariant derivatives, derive formulae for these operations in spherical polar coordinates. Compare your results with those in Jackson (1999) or an equivalent text. Are they identical? Should they be?