General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 06 - Solutions

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Problem 01

Defining *angles* between vectors X, Y as

$$X \sphericalangle Y := \frac{g(X,Y)}{\sqrt{g(X,X)} \cdot \sqrt{g(Y,Y)}}$$

(compare to definition in Euclidian & unitary spaces), leads to the preservation of angles by any conformal metric-transformation $g_{\mu\nu}\big|_x = \Omega(x) \cdot \widetilde{g}_{\mu\nu}\big|_x$, as

$$\frac{g(X,Y)}{\sqrt{g(X,X)} \cdot \sqrt{g(Y,Y)}} = \frac{\Omega \cdot \widetilde{g}(X,Y)}{\sqrt{\Omega \cdot \widetilde{g}(X,X)} \cdot \sqrt{\Omega \cdot \widetilde{g}(Y,Y)}}$$

Furthermore, for any vector X, $\widetilde{g}(X,X) = 0$ implies $\underbrace{g(X,X)}_{\Omega \cdot \widetilde{g}(X,X)} = 0$, that is, null-curves are still null-curves after conformal metric-transformations.

Problem 02

Starting with

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\varkappa} \left(\partial_\mu g_{\nu\varkappa} + \partial_\nu g_{\mu\varkappa} - \partial_\varkappa g_{\mu\nu} \right)$$

and a diagonal metric $g_{\mu\nu}$, we consider following cases:

1. $\mu \neq \nu \neq \lambda \neq \mu$: Then $g_{\mu\nu} = 0$ and $g^{\lambda\varkappa} = 0$ for $\lambda \neq \varkappa$, thus

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\lambda} \Big(\partial_\mu \underbrace{g_{\nu\lambda}}_{0} + \partial_\nu \underbrace{g_{\mu\lambda}}_{0} \Big) = 0$$

2. $\mu = \nu \neq \lambda$, then

$$\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2} \underbrace{g^{\lambda\varkappa}}_{\substack{0 \\ \text{for} \\ \varkappa \neq \lambda}} \left(2\partial_\mu g_{\mu\varkappa} - \partial_\varkappa g_{\mu\mu} \right) = \frac{1}{2} g^{\lambda\lambda} \Big(2\partial_\mu \underbrace{g_{\mu\lambda}}_{0} - \partial_\lambda g_{\mu\mu} \Big) = -\frac{1}{2} g^{\lambda\lambda} \partial_\lambda g_{\mu\mu} = -\frac{1}{2} \frac{1}{g_{\lambda\lambda}} \partial_\lambda g_{\mu\mu}$$

Note that the inverse $g^{\mu\nu}$ of the diagonal $g_{\mu\nu}$ is the diagonal matrix with entries $g^{\mu\mu} = 1/g_{\mu\mu}$.

3. $\nu = \lambda \neq \mu$, then

$$\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2} \underbrace{g^{\lambda\varkappa}}_{\substack{0 \\ \text{for} \\ \varkappa \neq \lambda}} \left(\partial_\mu g_{\lambda\varkappa} + \partial_\lambda g_{\mu\varkappa} - \partial_\varkappa g_{\mu\lambda} \right) = \frac{1}{2} g^{\lambda\lambda} \left(\partial_\mu g_{\lambda\lambda} + \cancel{\partial_\lambda g_{\mu\lambda}} - \cancel{\partial_\lambda g_{\mu\lambda}} \right) = \frac{1}{2} \underbrace{g^{\lambda\lambda}}_{1/g_{\lambda\lambda}} \partial_\mu g_{\lambda\lambda}$$

$$\overset{(\clubsuit)}{=} \frac{1}{2} \frac{1}{|g_{\lambda\lambda}|} \partial_\mu |g_{\lambda\lambda}| = \partial_\mu \left(\ln \sqrt{|g_{\lambda\lambda}|} \right)$$

(♣): Note that, as g is invertible and diagonal, all entries $g_{\lambda\lambda}$ are non-zero. As g is further smooth on M, it's entries are also non-zero with constant signum in some open neighborhood of the considered point, thus

$$\begin{aligned}\frac{\partial_\mu |g_{\lambda\lambda}|(\mathbf{x})}{|g_{\lambda\lambda}|(\mathbf{x})} &= \frac{1}{|g_{\lambda\lambda}(\mathbf{x})|} \cdot \lim_{h\to 0} \frac{|g_{\lambda\lambda}(\mathbf{x}+h\mathbf{e}_\mu)| - |g_{\lambda\lambda}(\mathbf{x})|}{h} \\ &= \frac{\operatorname{sgn}(g_{\lambda\lambda}(\mathbf{x}))}{\operatorname{sgn}(g_{\lambda\lambda}(\mathbf{x}))} \cdot \frac{1}{g_{\lambda\lambda}(\mathbf{x})} \cdot \lim_{h\to 0} \frac{g_{\lambda\lambda}(\mathbf{x}+h\mathbf{e}_\mu) - g_{\lambda\lambda}(\mathbf{x})}{h} \\ &= \frac{\partial_\mu g_{\lambda\lambda}}{g_{\lambda\lambda}}\end{aligned}$$

4. $\mu = \nu = \lambda$, then

$$\Gamma^\lambda_{\lambda\lambda} = \frac{1}{2} \underbrace{g^{\lambda\varkappa}}_{\substack{0\\ \text{for}\\ \varkappa\neq\lambda}} (\partial_\lambda g_{\lambda\varkappa} + \partial_\lambda g_{\lambda\varkappa} - \partial_\varkappa g_{\lambda\lambda}) = \frac{1}{2} g^{\lambda\lambda} \partial_\lambda g_{\lambda\lambda} = \frac{1}{2}\frac{\partial_\lambda g_{\lambda\lambda}}{g_{\lambda\lambda}} \overset{\substack{\text{similarly}\\ \text{to (3)}}}{=} \partial_\lambda \ln \sqrt{|g_{\lambda\lambda}|}$$

Problem 03

Beginning with the Euklidian metric in spherical coordinates

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \vartheta \end{pmatrix}$$

and using the results from (02) we write

$$\Gamma^\varphi_{r\vartheta} = \Gamma^\varphi_{\vartheta r} = \Gamma^\vartheta_{r\varphi} = \Gamma^\vartheta_{\varphi r} = \Gamma^r_{\vartheta\varphi} = \Gamma^r_{\varphi\vartheta} = 0$$

$$\Gamma^\vartheta_{rr} = -\frac{1}{2}\frac{1}{g_{\vartheta\vartheta}} \underbrace{\partial_\vartheta g_{rr}}_{0} = 0 \qquad \Gamma^\varphi_{rr} = -\frac{1}{2}\frac{1}{g_{\varphi\varphi}} \underbrace{\partial_\varphi g_{rr}}_{0} = 0$$

$$\Gamma^r_{\vartheta\vartheta} = -\frac{1}{2}\frac{1}{g_{rr}} \underbrace{\partial_r g_{\vartheta\vartheta}}_{2r} = -r \qquad \Gamma^\varphi_{\vartheta\vartheta} = -\frac{1}{2}\frac{1}{g_{\varphi\varphi}} \underbrace{\partial_\varphi g_{\vartheta\vartheta}}_{0} = 0$$

$$\Gamma^r_{\varphi\varphi} = -\frac{1}{2}\frac{1}{g_{rr}} \underbrace{\partial_r g_{\varphi\varphi}}_{2r\sin^2\vartheta} = -r\sin^2\vartheta \qquad \Gamma^\vartheta_{\varphi\varphi} = -\frac{1}{2}\frac{1}{g_{\vartheta\vartheta}} \underbrace{\partial_\vartheta g_{\varphi\varphi}}_{2r^2\sin\vartheta\cos\vartheta} = -\sin\vartheta\cos\vartheta$$

$$\Gamma^\vartheta_{r\vartheta} = \Gamma^\vartheta_{\vartheta r} = \frac{1}{2}\frac{1}{g_{\vartheta\vartheta}} \underbrace{\partial_r g_{\vartheta\vartheta}}_{2r} = \frac{1}{r} \qquad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{2}\frac{1}{g_{\varphi\varphi}} \underbrace{\partial_r g_{\varphi\varphi}}_{2r\sin^2\vartheta} = \frac{1}{r}$$

$$\Gamma^r_{\vartheta r} = \Gamma^r_{r\vartheta} = \frac{1}{2}\frac{1}{g_{rr}} \underbrace{\partial_\vartheta g_{rr}}_{0} = 0 \qquad \Gamma^\varphi_{\vartheta\varphi} = \frac{1}{2}\frac{1}{g_{\varphi\varphi}} \underbrace{\partial_\vartheta g_{\varphi\varphi}}_{2r^2\sin\vartheta\cos\vartheta} = \cot\vartheta$$

$$\Gamma^r_{\varphi r} = \Gamma^r_{r\varphi} = \frac{1}{2}\frac{1}{g_{rr}} \underbrace{\partial_\varphi g_{rr}}_{0} = 0 \qquad \Gamma^\vartheta_{\varphi\vartheta} = \Gamma^\vartheta_{\vartheta\varphi} = \frac{1}{2}\frac{1}{g_{\vartheta\vartheta}} \underbrace{\partial_\varphi g_{\vartheta\vartheta}}_{0} = 0$$

and finally

$$\Gamma^r_{rr} = \frac{1}{2}\frac{\partial r g_{rr}}{g_{rr}} = 0 \qquad \Gamma^\vartheta_{\vartheta\vartheta} = \frac{\partial_\vartheta g_{\vartheta\vartheta}}{g_{\vartheta\vartheta}} = 0 \qquad \Gamma^\varphi_{\varphi\varphi} = \frac{\partial_\varphi g_{\varphi\varphi}}{g_{\varphi\varphi}} = 0$$

Problem 04 (Carroll, Problem 3.2)

As is known, in coordinates for which all first derivatives of the components $g_{\mu\nu}$ vanish (Riemann-normal-coordinates), partial and covariant derivatives are identical, that is

$$\nabla_\mu V^\nu = \partial_\mu V^\nu = (\nabla_{\partial_\mu} V)^\nu$$

In any other coordinate system, the covariant derivative is given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\varkappa} V^\varkappa$$

As the tensor field defined by the covariant derivative does not depend on the coordinates chosen (in other words: ∇ creates tensors), it may be used to express fields, defined through partial derivatives in Riemann-normal-coordinates, in other coordinates as well. Thus for example in spherical coordinates:

$$\vec{\nabla}\Phi \overset{\text{cartesian}}{=} (\partial^\mu \Phi)\partial_\mu = g^{\mu\nu}(\partial_\nu\Phi)\partial_\mu \overset{\text{general}}{=} g^{\mu\nu}(\nabla_\nu\Phi)\partial_\mu \overset{\substack{\text{spherical}\\(03)}}{=} (\partial_r\Phi)\partial_r + \frac{1}{r^2}(\partial_\vartheta\Phi)\partial_\vartheta + \frac{1}{r^2\sin^2\vartheta}(\partial_\varphi\Phi)\partial_\varphi$$

$$\operatorname{div}\mathbf{V} \overset{\text{cartesian}}{=} \partial_\mu V^\mu = \nabla_\mu V^\mu \overset{\text{general}}{=} \partial_\mu V^\mu + \Gamma^\mu_{\mu\varkappa}V^\varkappa \overset{\substack{\text{spherical}\\(03)}}{=} \partial_r V^r + \partial_\vartheta V^\vartheta + \partial_\varphi V^\varphi + \frac{2}{r}V^r + \cot\vartheta V^\vartheta$$

$$\operatorname{rot}\mathbf{V} \overset{\text{cartesian}}{=} \partial_\mu V_\nu \epsilon^{\mu\nu\varkappa}\partial_\varkappa = \nabla_\mu V^\lambda g_{\lambda\nu}\epsilon^{\mu\nu\varkappa}\partial_\varkappa \overset{\text{general}}{=} \left(\partial_\mu V^\lambda + \Gamma^\lambda_{\mu\varkappa}V^\varkappa\right) g_{\lambda\nu}\operatorname{sgn}(\mu\nu\varkappa) \underbrace{\frac{1}{\sqrt{|\det g|}}}_{\substack{1/r^2\sin\vartheta\\ \text{for spherical}}} \cdot\partial_\varkappa$$

$$\overset{\substack{\text{spherical}\\(03)}}{=} \frac{1}{\sin\vartheta}\cdot\left[\sin^2\vartheta\cdot\partial_\vartheta V^\varphi - \partial_\varphi V^\vartheta + 2\sin\vartheta\cos\vartheta\cdot V^\varphi\right]\cdot\partial_r$$

$$+\frac{1}{r^2\sin\vartheta}\cdot\left[\partial_\varphi V^r - r^2\sin^2\vartheta\cdot\partial_r V^\varphi - 2r\sin^2\vartheta\cdot V^\varphi\right]\cdot\partial_\vartheta$$

$$+\frac{1}{r^2\sin\vartheta}\cdot\left[r^2\partial_r V^\vartheta - \partial_\vartheta V^r + 2rV^\vartheta\right]\cdot\partial_\varphi$$

for any vector-field

$$\mathbf{V} = V^r\partial_r + V^\vartheta\partial_\vartheta + V^\varphi\partial_\varphi$$

Notes:

(i) Typically the gradient of $\Phi \in \mathcal{F}$ is defined as the 1-form $d\Phi$ for which $d\Phi(X) = X\Phi$ for all vectors X:

$$d\Phi = \partial_\mu\Phi\ dx^\mu \overset{\text{spherical}}{=} \partial_r\Phi\ dr + \partial_\vartheta\Phi\ d\vartheta + \partial_\varphi\Phi\ d\varphi$$

As the vectorial gradient $\vec{\nabla}\Phi$ is defined as the vector field for which

$$g(\vec{\nabla}\Phi,\ \cdot) = d\Phi$$

it is given by

$$\vec{\nabla}\Phi = g^{\mu\nu}\partial_\mu\Phi\ \partial_\nu$$

(indices pulled up).

(ii) Typically, one finds in classical physics textbooks usage of the normalized coordinate-vectors $\mathbf{e}_\mu$ instead of the coordinate-vectors ∂_μ. Normalizing the basis $\mathbf{e}_\mu := \partial_\mu / \sqrt{g(\partial_\mu, \partial_\mu)}$ (non-coordinate!), that is,

$$\mathbf{e}_r := \partial_r \ , \ \mathbf{e}_\vartheta := \frac{1}{r}\partial_\vartheta \ , \ \mathbf{e}_\varphi := \frac{1}{r\sin\vartheta}\partial_\varphi$$

and thus

$$\mathbf{V} = V^\mu \partial_\mu = \underbrace{V^\mu \sqrt{g(\partial_\mu, \partial_\mu)}}_{\widetilde{V}^\mu} \cdot \mathbf{e}_\mu = \widetilde{V}^\mu \mathbf{e}_\mu$$

that is,

$$\widetilde{V}^r := V^r \ , \ \widetilde{V}^\vartheta := rV^\vartheta \ , \ \widetilde{V}^\varphi := r\sin\vartheta \cdot V^\varphi$$

yields

$$\vec{\nabla}\Phi = (\partial_r \Phi) \cdot \mathbf{e}_r + \frac{1}{r}(\partial_\vartheta \Phi) \cdot \mathbf{e}_\vartheta + \frac{1}{r\sin\vartheta}(\partial_\varphi \Phi) \cdot \mathbf{e}_\varphi$$

$$\operatorname{div} \mathbf{V} = \frac{1}{r^2}\partial_r(r^2 \widetilde{V}^r) + \frac{1}{r\sin\vartheta}\left[\partial_\vartheta(\sin\vartheta \cdot \widetilde{V}^\vartheta) + \partial_\varphi \widetilde{V}^\varphi\right]$$

$$\operatorname{rot} \mathbf{V} = \frac{1}{r\sin\vartheta}\left[\partial_\vartheta(\sin\vartheta \cdot \widetilde{V}^\varphi) - \partial_\varphi \widetilde{V}^\vartheta\right] \cdot \mathbf{e}_r + \frac{1}{r}\left[\frac{1}{\sin\vartheta}\partial_\varphi \widetilde{V}^r - \partial_r(r\widetilde{V}^\varphi)\right] \cdot \mathbf{e}_\vartheta + \frac{1}{r}\left[\partial_r(r\widetilde{V}^\vartheta) - \partial_\vartheta \widetilde{V}^r\right] \cdot \mathbf{e}_\varphi$$

Comparing the latest form with [1] verifies the correctness of the results.

References

[1] *Taschenbuch Mathematischer Formeln*, H.J. Bartsch
Fachbuchverlag Leipzig, 2004