General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 05 - Solutions

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Problem 01 (Carroll, Problem 2.4)

Let $\mathbf{u} : U \subseteq M \to \mathbb{R}^n,\ p \mapsto (u^1(p), .., u^n(p))$ be the considered chart. Beginning with the defining properties of vector fields we write

$$[X,Y](af+bg) \overset{\text{def}}{=} XY(af+bg) - YX(af+bg) = aXYf + bXYg - aYXf - bYXg$$

$$= a(XY-YX)f + b(XY-YX)g = a[X,Y]f + b[Y,X]g$$

$$[X,Y](fg) = XY(fg) - YX(fg) = X(fYg + gYf) - Y(fXg + gXf)$$

$$= \cancel{(Xf)(Yg)} + fXYg + \cancel{(Xg)(Yf)} + gXYf - \cancel{(Yf)(Xg)} - fYXg - \cancel{(Yg)(Xf)} - gYXf$$

$$= f(XY-YX)g + g(XY-YX)f = f[X,Y]g + g[X,Y]f$$

Finally, using $X = X^\nu \partial_\nu$ & $Y = Y^\nu \partial_\nu$ we write

$$[X,Y]^\mu = [X,Y](u^\mu) = XYu^\mu - YXu^\mu = X\big(Y^\nu \underbrace{\partial_\nu u^\mu}_{\delta^\mu_\nu}\big) - Y\big(X^\nu \underbrace{\partial_\nu u^\mu}_{\delta^\mu_\nu}\big) = XY^\mu - YX^\mu$$

$$= X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu$$

Problem 02 (Carroll, Problem 2.7)

The plane $\{y = 0\}$ is equivalent to $\{\varphi = 0\}$, on which

$$x = \sinh\chi \sin\vartheta$$
$$z = \cosh\chi \cos\vartheta$$

(a) The transformation matrix between the two coordinate systems is given by

$$\frac{\partial(x,z)}{\partial(\chi,\vartheta)} = \begin{pmatrix} \cosh\chi\sin\vartheta & \sinh\chi\cos\vartheta \\ \sinh\chi\cos\vartheta & -\cosh\chi\sin\vartheta \end{pmatrix}$$

with inverse

$$\frac{\partial(\chi,\vartheta)}{\partial(x,z)} = \frac{1}{\left|\frac{\partial(x,z)}{\partial(\chi,\vartheta)}\right|} \begin{pmatrix} -\cosh\chi\sin\vartheta & -\sinh\chi\cos\vartheta \\ -\sinh\chi\cos\vartheta & \cosh\chi\sin\vartheta \end{pmatrix} = \frac{1}{\cosh^2\chi - \cos^2\vartheta} \cdot \frac{\partial(x,z)}{\partial(\chi,\vartheta)} \tag{0.1}$$

(b) In coordinates (x,z) the metric is given by

$$g = dx \otimes dx + dz \otimes dz \quad \leftrightarrow \quad (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Transforming to (χ,ϑ)-coordinates yields the representation

$$(g_{i'j'}) = \underbrace{\left(\frac{\partial(x,z)}{\partial(\chi,\vartheta)}\right)^T}_{\frac{\partial(x,z)}{\partial(\chi,\vartheta)}} \underbrace{(g_{ij})}_{1_{2\times 2}} \left(\frac{\partial(x,z)}{\partial(\chi,\vartheta)}\right) \overset{(0.1)}{=} (\cosh^2\chi - \cos^2\vartheta)\cdot 1_{2\times 2}$$

that is

$$g = (\cosh^2\chi - \cos^2\vartheta)(d\chi \otimes d\chi + d\vartheta \otimes d\vartheta)$$

Problem 03 (Carroll, problem 2.8)

Let $dx^1,..,dx^n$ be the dual basis to the coordinate vectors $\partial_1,..,\partial_n$ and $\mathcal{F}$ the set of all smooth, real functions on the manifold M. Then due to the $\mathcal{F}$-linearity of the wedge product and $\mathbb{R}$-linearity of the exterior derivative, it suffices to show the statement for p & q-forms of the kind[1]

$$\omega = f\cdot dx^{i_1}\wedge\cdots\wedge dx^{i_p} \quad , \quad \eta = g\cdot dx^{i_{p+1}}\wedge\cdots\wedge dx^{i_{p+q}} \quad , \quad f,g\in\mathcal{F}$$

as any p form a can be written as

$$a = h_{j_1\ldots j_p} dx^{j_1}\wedge\cdots\wedge dx^{j_p} \quad , \quad h_{j_1\ldots j_p}\in\mathcal{F}$$

Beginning with

$$da := \frac{1}{p!}(\partial_\mu a_{j_1..j_p})\cdot dx^\mu \wedge dx^{j_1}\ldots dx^{j_p}$$

for any p-Form a, we write for $a = h\cdot dx^{i_1}\wedge\cdots\wedge dx^{i_p}$:

$$d\left(h\cdot dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right) = \frac{1}{p!}\partial_\mu \underbrace{\left(h\cdot dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)_{j_1..j_p}}_{h\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)\delta^{i_{\sigma(1)}}_{j_1}\cdot..\cdot\delta^{i_{\sigma(p)}}_{j_p}} dx^\mu\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p}$$

$$= \frac{\partial_\mu h}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)\cdot dx^\mu\wedge \underbrace{dx^{i_{\sigma(1)}}\wedge\cdots\wedge dx^{i_{\sigma(p)}}}_{\operatorname{sgn}(\sigma)\cdot dx^{i_1}\wedge\cdots\wedge dx^{i_p}} = \frac{\partial_\mu h}{p!}\cdot p!\cdot dx^\mu\wedge dx^{i_1}\wedge..\wedge dx^{i_{p+q}}$$

$$= (\partial_\mu h)\cdot dx^\mu\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

[1] If the statement holds for $d(\omega_i\wedge\eta_j)$ with ω_i and η_j p- and q-forms respectively, then it follows

$$d\left[\sum_i\omega_i\wedge\sum_j\eta_j\right] = \sum_{i,j}d(\omega_i\wedge\eta_i) = \sum_{i,j}d\omega_i\wedge\eta_j + (-1)^p\sum_{i,j}\omega_i\wedge d\eta_j$$

$$= \left(d\sum_i\omega_i\right)\wedge\left(\sum_j\eta_j\right) + (-1)^p\left(\sum_i\omega^i\right)\wedge\left(d\sum_j\eta_j\right)$$

Thus finally

$$
\begin{aligned}
d\,(\omega \wedge \eta) &= \partial_\mu (fg) \cdot dx^\mu \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p+q}} \\
&= \underbrace{(\partial_\mu f) \cdot dx^\mu \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}}_{d\omega} \wedge \underbrace{g dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_{p+q}}}_{\eta} \\
&+ (-1)^p \cdot \underbrace{f \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_p}}_{\omega} \wedge \underbrace{(\partial_\mu g) \cdot dx^\mu \wedge dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_{p+q}}}_{d\eta} \\
&= d\omega \wedge \eta + (-1)^p \cdot \omega \wedge d\eta
\end{aligned}
$$

Note that the in associativity and antisymmetry of the wedge-product has been used above.
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