General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 04

November 15, 2009

Problem 01 (Carroll, Problem 2.1)

Just because a manifold is topologically nontrivial doesn't necessarily mean it can't be covered with a single chart. In contrast to the circle S^1, show that the infinite cylinder $\mathbb{R} \times S^1$ can be covered with just one chart, by explicitly constructing the map.

Problem 02 (Carroll, Problem 2.3)

Show that the two-dimensional Torus T^2 is a manifold, by explicitly constructing an appropriate atlas. (Not a maximal one, obviously.)

Problem 03

If one believes that esthetics should be an important consideration in physical laws, then by symmetry Maxwell's laws should read

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^{\mu}$$

$$*F^{\mu\nu}{}_{,\nu} = 4\pi K^{\mu}$$

What would the significance of K^{μ} be?

Problem 04

Let t, x, y, z be Lorentz coordinates in flat space-time, and let

$$r = (x^2 + y^2 + z^2)^{1/2}, \quad \vartheta = \arccos(z/r), \quad \varphi = \arctan(y/x)$$

be the corresponding spherical coordinates. Then

$$\mathbf{e}_0 = \frac{\partial \mathcal{P}}{\partial t}, \ \mathbf{e}_r = \frac{\partial \mathcal{P}}{\partial r}, \ \mathbf{e}_\vartheta = \frac{\partial \mathcal{P}}{\partial \vartheta}, \ \mathbf{e}_\varphi = \frac{\partial \mathcal{P}}{\partial \varphi}$$

is a coordinate basis, and

$$\mathbf{e}_{\hat{0}} = \frac{\partial \mathcal{P}}{\partial t}, \ \mathbf{e}_{\hat{r}} = \frac{\partial \mathcal{P}}{\partial r}, \ \mathbf{e}_{\hat{\vartheta}} = \frac{1}{r}\frac{\partial \mathcal{P}}{\partial \vartheta}, \ \mathbf{e}_{\hat{\varphi}} = \frac{1}{r\sin\vartheta}\frac{\partial \mathcal{P}}{\partial \varphi}$$

is a noncoordinate basis.

(a) Draw a picture of $\mathbf{e}_\vartheta, \mathbf{e}_\varphi, \mathbf{e}_{\hat{\vartheta}}$ and $\mathbf{e}_{\hat{\varphi}}$ at several different points on a sphere of constant t, r.

(b) What are the 1-form bases $\{\omega^\alpha\}$ and $\{\omega^{\hat{\alpha}}\}$ dual to these tangent-vector bases?

(c) What is the transformation matrix linking the original Lorentz frame to the spherical coordinate frame $\{\mathbf{e}_\alpha\}$?

(d) Use this transformation matrix to calculate the metric components $g_{\alpha\beta}$ in the spherical coordinate basis and invert the resulting matrix to get $g^{\alpha\beta}$.

(e) Show that the noncoordinate basis $\{\mathbf{e}_{\hat{\alpha}}\}$ is orthonormal everywhere.

(f) Write the gradient of a function f in terms of the spherical coordinate and noncoordinate bases.

(g) What are the components of the Levi-Civita tensor in the spherical coordinate and noncoordinate bases?