General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 04 - Solutions

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Problem 01 (Carroll, Problem 2.1)

Set $U := \underbrace{\mathbb{R}^2 \setminus \{0\}}_{\text{open}}$ and let $\varphi : S^1 \to [0, 2\pi)$ be the natural identification of points in S^1 with their respective *angle*[1]. Then the map

$$\Phi : \mathbb{R} \times S^1 \to U \quad , \quad \Phi(r, s) = (e^r \cos\varphi(s),\ e^r \sin\varphi(s)) \quad , \quad r \in \mathbb{R},\ s \in S^1$$

is a chart covering the whole $\mathbb{R} \times S^1$.

Problem 02 (Carroll, Problem 2.3)

Let $S^1 \hookrightarrow \mathbb{R}^2$ and each point $s \in S^1$ be identified with it's natural angle $\varphi(s) \in [0, 2\pi)$. First consider the open covering of S^1

$$S^1 = \underbrace{\{(x,y) \in S^1 : y > 0\}}_{U_1} \cup \underbrace{\{(x,y) \in S^1 : x < 0\}}_{U_2} \cup \underbrace{\{(x,y) \in S^1 : y < 0\}}_{U_3} \cup \underbrace{\{(x,y) \in S^1 : x > 0\}}_{U_4}$$

and the bijective, smooth mappings

$$\Phi_i : U_i \to \varphi(U_i) \quad , \quad \Phi_i(s) = \varphi(s) \quad , \quad s \in U_i,\ i = 1, .., 4$$

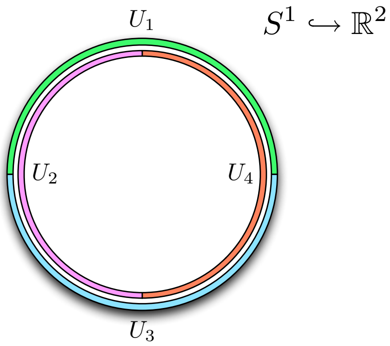

Figure 0.1: On problem 02: Covering of S^1 by open semi-circles.

Then $U_i \times U_j,\ i, j = 1, .., 4$ form an open covering of the torus $T^2 := S^1 \times S^1$. Moreover, the maps

$$\Phi_{i,j} : U_i \times U_j \to \varphi(U_i) \times \varphi(U_j) \quad , \quad \Phi_{i,j}(s_1, s_2) = (\varphi(s_1),\ \varphi(s_2)) \quad , \quad s_1 \in U_i,\ s_2 \in U_j,\ i, j = 1, .., 4$$

are all bijective, smooth, compatible and form an atlas of T^2.

[1] Defined for $s \in S^1$ as the angle $\varphi \in [0, 2\pi)$ for which $s = (\cos\varphi, \sin\varphi)$ (S^1 naturally embedded in $\mathbb{R}^2$).

Problem 03

Note

For completely antisymmetric $X^{\mu_1\cdots\mu_n}$ and arbitrary $Y_{\nu_1\ldots\nu_n}$ the relation

$$X^{\mu_1\cdots\mu_n}Y_{\mu_1\ldots\mu_n} = X^{\mu_1\cdots\mu_n}Y_{[\mu_1\ldots\mu_n]} \tag{0.1}$$

holds.

Proof: By definition

$$\begin{aligned}
X^{\mu_1\cdots\mu_n}Y_{[\mu_1\ldots\mu_n]} &= X^{\mu_1\cdots\mu_n}\frac{1}{n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\cdot Y_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}} \\
&= X^{\mu_{\sigma^{-1}(1)}\cdots\mu_{\sigma^{-1}(n)}}\frac{1}{n!}\sum_{\sigma\in S_n}\underbrace{\operatorname{sgn}(\sigma)}_{\operatorname{sgn}(\sigma^{-1})}\cdot Y_{\mu_1\ldots\mu_n} \\
&= X^{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}\frac{1}{n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)Y_{\mu_1\ldots\mu_n} \\
&= \underbrace{X^{[\mu_1\cdots\mu_n]}}_{X^{\mu_1\cdots\mu_n}}Y_{\mu_1\ldots\mu_n} = X^{\mu_1\cdots\mu_n}Y_{\mu_1\ldots\mu_n}
\end{aligned}$$

$\square$

Variant 1

We shall assume the coordinates to be such that $g_{\mu\nu} = \operatorname{diag}(-1,1,1,1)$. Let $\varepsilon_i := g(\partial_i,\partial_i) \in \{\pm 1\}$, then

$$*(dx^{i_1}\wedge\cdots\wedge dx^{i_p}) = \varepsilon_{i_1}\cdot ..\cdot\varepsilon_{i_p}\cdot dx^{i_{p+1}}\wedge\cdots\wedge dx^{i_n}$$

(no summation!) with $(i_1,\ldots,i_n)$ an even permutation of $(0,1,..,n)$. In particular

$$\begin{aligned}
*(dx^0\wedge dx^1) &= -dx^2\wedge dx^3 \\
*(dx^0\wedge dx^2) &= dx^1\wedge dx^3 \\
*(dx^0\wedge dx^3) &= -dx^1\wedge dx^2 \\
*(dx^1\wedge dx^2) &= dx^0\wedge dx^3 \\
*(dx^1\wedge dx^3) &= -dx^0\wedge dx^2 \\
*(dx^2\wedge dx^3) &= dx^0\wedge dx^1
\end{aligned}$$

With

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

that is,

$$\begin{aligned}
F = &- E_1dx^0\wedge dx^1 - E_2dx^0\wedge dx^2 - E_3dx^0\wedge dx^3 \\
&+ B_3dx^1\wedge dx^2 - B_2dx^1\wedge dx^3 + B_1dx^2\wedge dx^3
\end{aligned}$$

we can write

$$\begin{aligned}
*F = &- E_1 * (dx^0\wedge dx^1) - E_2 * (dx^0\wedge dx^2) - E_3 * (dx^0\wedge dx^3) \\
&+ B_3 * (dx^1\wedge dx^2) - B_2 * (dx^1\wedge dx^3) + B_1 * (dx^2\wedge dx^3) \\
= &E_1dx^2\wedge dx^3 + E_2dx^3\wedge dx^1 + E_3dx^1\wedge dx^2 \\
&+ B_3dx^0\wedge dx^3 + B_2dx^0\wedge dx^2 + B_1dx^0\wedge dx^1
\end{aligned}$$

or in components:

$$(*F)_{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

Contracting with g leads to

$$(*F)^{\mu\nu} \overset{\text{def}}{=} g^{\mu\rho} g^{\nu\sigma} (*F)_{\rho\sigma} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

Furthermore

$$\underbrace{\partial_\nu (*F)^{\mu\nu}}_{=:4\pi K^\mu} = \begin{pmatrix} -\partial_0 B_1 - \partial_0 B_2 - \partial_0 B_3 \\ \partial_0 B_1 + \partial_2 E_3 - \partial_3 E_2 \\ \partial_0 B_2 - \partial_1 E_3 + \partial_3 E_1 \\ \partial_0 B_3 + \partial_1 E_2 - \partial_2 E_1 \end{pmatrix} = \begin{pmatrix} -\operatorname{div} \mathbf{B} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \partial_0 \mathbf{B} + \nabla \times \mathbf{E} \end{pmatrix}$$

Thus, Maxwell's equations $\operatorname{div} \mathbf{B} = 0,\ \partial_0 \mathbf{B} = -\nabla \times \mathbf{E}$ are equivalent to $K^\mu = 0$.

Variant 2

The 1-form $*dF$ is given by:

$$(*dF)^\mu = (*dF)_\varkappa g^{\varkappa\mu} = \frac{1}{3} \varepsilon_{\rho\sigma\tau\varkappa} (dF)^{\rho\sigma\tau} g^{\varkappa\mu} = \frac{1}{3} \varepsilon_{\rho\sigma\tau\varkappa} g^{\nu\rho} g^{\beta\sigma} g^{\gamma\tau} g^{\varkappa\mu} (dF)_{\nu\beta\gamma}$$

$$= \frac{1}{3} \varepsilon_{\rho\sigma\tau\varkappa} g^{\nu\rho} g^{\beta\sigma} g^{\gamma\tau} g^{\varkappa\mu} \cdot 3\partial_{[\nu} F_{\beta\gamma]} = \varepsilon^{\nu\beta\gamma\mu} \partial_{[\nu} F_{\beta\gamma]} \overset{(0.1)}{=} \varepsilon^{\nu\beta\gamma\mu} \partial_\nu F_{\beta\gamma}$$

On the other hand, $\partial_\nu (*F)^{\mu\nu}$ is given by

$$\partial_\nu (*F)^{\mu\nu} = \partial_\nu (*F)_{\varkappa\lambda} g^{\mu\varkappa} g^{\nu\lambda} = \partial_\nu \left[\varepsilon_{\beta\gamma\varkappa\lambda} F^{\beta\gamma} \right] g^{\mu\varkappa} g^{\nu\lambda} = \underbrace{\varepsilon^{\beta\gamma\mu\nu}}_{-\varepsilon^{\nu\beta\gamma\mu}} \partial_\nu F_{\beta\gamma} = -\varepsilon^{\nu\beta\gamma\mu} \partial_\nu F_{\beta\gamma}$$

Thus:

$$(*dF)^\mu = -\partial_\nu (*F)^{\mu\nu} =: 4\pi K^\mu$$

But as is known, Maxwell's equations imply $dF = 0$, thus $K^\mu = 0$.

Problem 04

Note: Spherical & cartesian coordinates relate through

$$\mathbf{x} := \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ r \sin\vartheta \cos\varphi \\ r \sin\vartheta \sin\varphi \\ r \cos\vartheta \end{pmatrix}$$

Notation: Let

$$u^0 := t, u^1 = \rho, u^2 = \vartheta, u^3 = \varphi$$

and

$$x^0 = t, x^1 = x, x^2 = y, x^3 = z$$

(a) The coordinate vectors $\partial_r, \partial_\vartheta, \partial_\varphi, \partial_t$ are in cartesian representation given by

$$\partial_r = \frac{\partial \mathbf{x}}{\partial r} = \begin{pmatrix} 0 \\ \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \quad , \quad \partial_\vartheta = \frac{\partial \mathbf{x}}{\partial \vartheta} = \begin{pmatrix} 0 \\ r\cos\vartheta\cos\varphi \\ r\cos\vartheta\sin\varphi \\ -r\sin\vartheta \end{pmatrix}$$

$$\partial_\varphi = \frac{\partial \mathbf{x}}{\partial \varphi} = \begin{pmatrix} 0 \\ -r\sin\vartheta\sin\varphi \\ r\sin\vartheta\cos\varphi \\ 0 \end{pmatrix} \quad , \quad \partial_t = \frac{\partial \mathbf{x}}{\partial t} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

With respect to the semi-metric (cartesian)

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{0.2}$$

their *norms* are given by

$$g(\partial_t, \partial_t) = -1 \quad , \quad g(\partial_r, \partial_r) = 1 \quad , \quad g(\partial_\vartheta, \partial_\vartheta) = r^2 \quad , \quad g(\partial_\varphi, \partial_\varphi) = r^2 \sin^2\vartheta \tag{0.3}$$

so that the modified basis vectors

$$\widetilde{\partial}_t := \partial_t \quad , \quad \widetilde{\partial}_r := \partial_r \quad , \quad \widetilde{\partial}_\vartheta := \frac{1}{r}\partial_\vartheta \quad , \quad \widetilde{\partial}_\varphi := \frac{1}{r\sin\vartheta}\partial_\varphi \tag{0.4}$$

are actually normalized[2]. Fig. (0.2) shows the vectors $\partial_\vartheta, \partial_\varphi, \widetilde{\partial}_\vartheta, \widetilde{\partial}_\varphi$ at typical points on S^2.

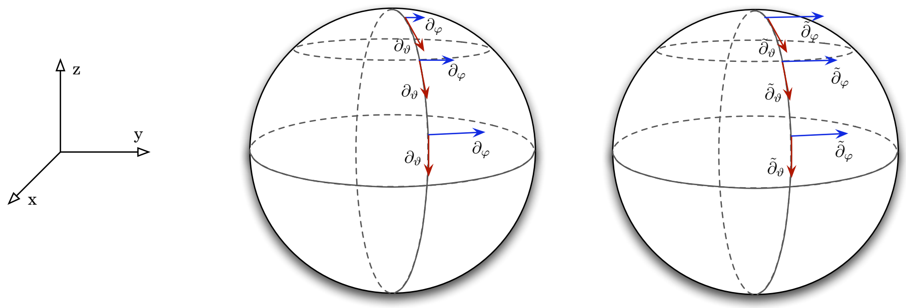

Figure 0.2: On problem 04 (a): $\partial_\vartheta, \partial_\varphi, \widetilde{\partial}_\vartheta, \widetilde{\partial}_\varphi$ on a sphere of constant $r = 1,\ t = 0$.

(b) The dual base $\{dt, dr, d\vartheta, d\varphi\}$ corresponding to $\{\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi\}$ is defined through

$$du^\alpha(\partial_\beta) = \delta_{\alpha\beta} \tag{0.5}$$

[2] Note that the restriction of g on the tangent-space of the sub-mannifold $\{t : \text{const}\}$ is identical to the Euclidian one.

In cartesian representation these covectors are given by $du^\alpha = \nabla_{\mathbf{x}} u^\alpha$, that is:

$$dt = \frac{\partial t}{\partial \mathbf{x}} = (1, 0, 0, 0)$$

$$dr = \frac{\partial r}{\partial \mathbf{x}} = \left(0, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = (0,\ \sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi,\ \cos\vartheta)$$

$$d\vartheta = \frac{\partial \vartheta}{\partial \mathbf{x}} = \left(0, \frac{1}{r}\cos\vartheta\cos\varphi,\ \frac{1}{r}\cos\vartheta\sin\varphi,\ -\frac{1}{r}\sin\vartheta\right)$$

$$d\varphi = \frac{\partial \varphi}{\partial \mathbf{x}} = \left(0,\ -\frac{1}{r}\frac{\sin\varphi}{\sin\vartheta},\ \frac{1}{r}\frac{\cos\varphi}{\sin\vartheta},\ 0\right)$$

Consequently, the covector-basis

$$\widetilde{dt} := dt,\ \widetilde{dr} := dr,\ \widetilde{d\vartheta} := r \cdot d\vartheta,\ \widetilde{d\varphi} := r\sin\vartheta \cdot d\varphi$$

is dual to the normalized vectors $\widetilde{\partial}_t, \widetilde{\partial}_r, \widetilde{\partial}_\vartheta, \widetilde{\partial}_\varphi$.

(c) The transformation matrix between the two vector-bases $\mathbf{e}_t, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ and $\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi$ is given by

$$\Lambda = \frac{\partial(t, r, \vartheta, \varphi)}{\partial(t, x, y, z)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi & \cos\vartheta \\ 0 & \frac{1}{r}\cos\vartheta\cos\varphi & \frac{1}{r}\cos\vartheta\sin\varphi & -\frac{1}{r}\sin\vartheta \\ 0 & -\frac{1}{r}\frac{\sin\varphi}{\sin\vartheta} & \frac{1}{r}\frac{\cos\varphi}{\sin\vartheta} & 0 \end{pmatrix} \tag{0.6}$$

with inverse

$$\Lambda^{-1} = \frac{\partial(t, x, y, z)}{\partial(t, r, \vartheta, \varphi)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ 0 & \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ 0 & \cos\vartheta & -r\sin\vartheta & 0 \end{pmatrix} \tag{0.7}$$

(d) In spherical coordinates the metric is given by

$$g_{\alpha\beta} = \underbrace{g_{\mu\nu}}_{\text{cartesian}} \frac{\partial x^\mu}{\partial u^\alpha}\frac{\partial x^\nu}{\partial u^\beta} \cong (\Lambda^{-1})^T g \Lambda^{-1} \overset{(0.7)}{=} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\vartheta \end{pmatrix} \tag{0.8}$$

with inverse

$$g^{\alpha\beta} \cong \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\vartheta} \end{pmatrix} \tag{0.9}$$

Furthermore, in the modified basis $\{\widetilde{\partial}_\alpha\}$ it is given by

$$g(\widetilde{\partial}_\alpha, \widetilde{\partial}_\beta) \cong \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{0.10}$$

(e) As $\partial_r, \partial_\vartheta, \partial_\varphi$ are also vectors in the tangent-plane of the sub-manifold $\{t = \text{const}\}$, their *scalar-product* $g(\partial_\alpha, \partial_\beta)$ is the same as their Euklidian one in cartesian representation. As is known, they form an orthogonal basis in $\mathbb{R}^3$, so that

$$g(\partial_r, \partial_\vartheta) = g(\partial_r, \partial_\varphi) = g(\partial_\vartheta, \partial_\varphi) = 0$$

Furthermore, as can be seen from their cartesian representation in part (a), all 3 are orthogonal to ∂_t (with respect to g). Thus $\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi$ are orthogonal with respect to g. Furthermore, using (0.3), it is obvious that the modified vectors

$$\widetilde{\partial}_\alpha = \frac{1}{\sqrt{|g(\partial_\alpha, \partial_\alpha)|}} \cdot \partial_\alpha$$

(compare to (0.4)) are normalized with respect to g, thus form an orthonormal basis.
Note: Alternatively, applying the 2-form g in (0.2) on $\{\widetilde{\partial}_\alpha\}$ directly leads to the same conclusions.

(f) The *gradient* of a scalar field f is defined as the vector field ∇f, which fulfills:

$$g(\nabla f, X) = Xf \qquad \forall\ X\ :\ \text{vector field}$$

that is,

$$\nabla f = g^{\alpha\beta}(\partial_\alpha f)\partial_\beta$$

In spherical & modified spherical coordinates thus

$$\nabla f \overset{(0.9)}{=} (\partial_t f) \cdot \partial_t + (\partial_r f) \cdot \partial_r + \frac{1}{r^2} \cdot (\partial_\vartheta f) \cdot \partial_\vartheta + \frac{1}{r^2 \sin^2 \vartheta} \cdot (\partial_\varphi f) \cdot \partial_\varphi$$

$$= (\partial_t f) \cdot \widetilde{\partial}_t + (\partial_r f) \cdot \widetilde{\partial}_r + \frac{1}{r} \cdot (\partial_\vartheta f) \cdot \widetilde{\partial}_\vartheta + \frac{1}{r \sin \vartheta} \cdot (\partial_\varphi f) \cdot \widetilde{\partial}_\varphi$$

(g) Beginning with the definition

$$\varepsilon_{\alpha_1 \dots \alpha_n} = \text{sgn}(\alpha_1, .., \alpha_n) \cdot \sqrt{\det(g_{\alpha\beta})}$$

of the Levi-Chivita tensor, whereas

$$\text{sgn}(\alpha_1, .., \alpha_n) := \begin{cases} \text{sgn}\left(\begin{pmatrix} 1 & \dots & n \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}\right) & : \alpha_i \neq \alpha_{j \neq i} \\ \\ 0 & : \text{otherwise} \end{cases}$$

and

$$\sqrt{\det(g(\partial_\alpha, \partial_\beta))} \overset{(0.8)}{=} r^2 \sin\vartheta \quad , \quad \sqrt{\det(g(\widetilde{\partial}_\alpha, \widetilde{\partial}_\beta))} \overset{(0.10)}{=} 1 \quad ,$$

we get

$$\varepsilon_{\alpha_1 .. \alpha_4} = \begin{cases} r^2 \sin\vartheta & : (\alpha_1, .., \alpha_4) \text{ even permutation of } (t, \rho, \vartheta, \varphi) \\ -r^2 \sin\vartheta & : (\alpha_1, .., \alpha_4) \text{ odd permutation of } (t, \rho, \vartheta, \varphi) \\ 0 & : \text{sonst} \end{cases}$$

in spherical coordinates and

$$\varepsilon_{\alpha_1 .. \alpha_4} = \begin{cases} 1 & : (\alpha_1, .., \alpha_4) \text{ even permutation of } (t, \rho, \vartheta, \varphi) \\ -1 & : (\alpha_1, .., \alpha_4) \text{ odd permutation of } (t, \rho, \vartheta, \varphi) \\ 0 & : \text{sonst} \end{cases}$$

in the non-coordinate basis.