General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 03

February 10, 2010

Problem 01 (Carroll, Problem 1.11)

Verify that $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$ is indeed equivalent to $\partial_{[\mu} F_{\nu\lambda]} = 0$, and they are both equivalent to the equations $\varepsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0$ and $\partial_i B^i = 0$.

Problem 02

Consider the two field theories we explicitly discussed, Maxwell's electromagnetism ($\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu$ and let $J^\mu = 0$) and the scalar field theory defined by $\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$.

- Express the components of the energy-momentum tensors of each theory in three-vector notation, using divergence, gradient, curl, electric and magnetic fields, and an overdot to denote time-derivativs.
  Note:

$$T^{\mu\nu}_{\text{scalar}} = \eta^{\mu\lambda}\eta^{\nu\sigma}(\partial_\lambda\phi)(\partial_\sigma\phi) - \eta^{\mu\nu}\left[\frac{1}{2}\eta^{\lambda\sigma}(\partial_\lambda\phi)(\partial_\sigma\phi) + V(\phi)\right]$$

$$T^{\mu\nu}_{\text{EM}} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}$$

- Using the equations of motion, verify (in any notation you like) that the energy-momentum tensors are conserved, that is $\partial_\nu T^{\mu\nu} = 0$. Note that:

$$\Box\phi - \frac{dV}{d\phi} = 0$$

$$\partial_\mu F^{\mu\nu} = J^\nu$$