General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 03 - Solutions

Stilianos Louca

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Problem 01 (Carroll, Problem 1.11)

Show that:

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \quad \forall\ \mu,\nu,\lambda \tag{0.1}$$

ist equivalent to $\partial_{[\mu}F_{\nu\lambda]} = 0$.
Indeed, this follows directly from

$$\partial_{[\mu}F_{\nu\lambda]} \overset{\text{def}}{=} \partial_\mu F_{\nu\lambda} - \partial_\mu \underbrace{F_{\lambda\nu}}_{-F_{\nu\lambda}} + \partial_\nu F_{\lambda\mu} - \partial_\nu \underbrace{F_{\mu\lambda}}_{-F_{\lambda\mu}} + \partial_\lambda F_{\mu\nu} - \partial_\lambda \underbrace{F_{\nu\mu}}_{-F_{\mu\nu}} = 2\big(\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu}\big)$$

Show that: Statement (0.1) is equivalent to

$$\varepsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0 \tag{0.2}$$

and

$$\partial_i B^i = 0 \tag{0.3}$$

Beginning with the Definition

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

Direction "$\Rightarrow$": Setting $\mu = 1, \nu = 2, \lambda = 3$ leads to

$$0 = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 \tag{0.4}$$

Moreover, with $B^i = \frac{1}{2}F_{jk}\varepsilon^{ijk}$ we can write

$$\varepsilon^{ijk}\partial_j \underbrace{E_k}_{F_{k0}} + \partial_0 B^i = \varepsilon^{ijk}\partial_j F_{k0} + \frac{1}{2}\varepsilon^{ijk}\partial_0 F_{jk} \overset{(0.1)}{=} \varepsilon^{ijk}\left[\partial_j F_{k0} - \frac{1}{2}\partial_j F_{k0} - \frac{1}{2}\partial_k \underbrace{F_{0j}}_{-F_{j0}}\right]$$

$$= \frac{\overbrace{\varepsilon^{ijk}}^{\substack{\text{antisymmetric} \\ \text{in } j,k}}}{2} \cdot \underbrace{[\partial_j F_{k0} + \partial_k F_{j0}]}_{\substack{\text{symmetric} \\ \text{in } j,k}} = 0 \quad \forall\ 1,2,3$$

Direction "$\Leftarrow$": Statement (0.1) is obviously invariant under any permutation[1] of λ, μ, ν. Moreover, it is trivial if two or more indices are equal (e.g. $\lambda = \nu$), since

$$\partial_\mu \underbrace{F_{\nu\lambda}}_{F_{\nu\nu}=0} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = \partial_\nu F_{\nu\mu} + \partial_\nu \underbrace{F_{\mu\nu}}_{-F_{\nu\mu}} = 0$$

Thus, statement (0.1) is practically dependent only on the set $I := \{\mu, \nu, \lambda\} \subseteq \{0, .., 3\}$ chosen.

Case $I = \{1, 2, 3\}$**:**

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 = 0$$

Case $I = \{0, \nu, \lambda\}\,,\ 1 \leq \nu, \lambda \leq 3,\ \nu \neq \lambda$**:** Let $i \in \{1, 2, 3\} \setminus \{\nu, \lambda\}$. Then by statement (0.2):

$$0 \overset{(0.2)}{=} \underbrace{\varepsilon^{ijk}}_{\substack{0 \text{ falls} \\ \{j,k\} \neq \{\nu,\lambda\}}} \partial_j F_{k0} + \frac{1}{2} \underbrace{\varepsilon^{ijk}}_{\substack{0 \text{ falls} \\ \{j,k\} \neq \{\nu,\lambda\}}} \partial_0 F_{jk} = \varepsilon^{i\nu\lambda} \partial_\nu F_{\lambda 0} + \underbrace{\varepsilon^{i\lambda\nu}}_{-\varepsilon^{i\nu\lambda}} \partial_\lambda \underbrace{F_{\nu 0}}_{-F_{0\nu}} + \frac{1}{2} \varepsilon^{i\nu\lambda} \partial_0 F_{\nu\lambda} + \frac{1}{2} \underbrace{\varepsilon^{i\lambda\nu}}_{-\varepsilon^{i\nu\lambda}} \partial_0 \underbrace{F_{\lambda\nu}}_{-F_{\nu\lambda}}$$

$$= \underbrace{\varepsilon^{i\nu\lambda}}_{\in\{\pm 1\}} \left[\partial_\nu F_{\lambda 0} + \partial_\lambda F_{0\nu} + \partial_0 F_{\nu\lambda}\right]$$

and thus

$$\partial_0 F_{\nu\lambda} + \partial_\nu F_{\lambda 0} + \partial_\lambda F_{0\nu} = 0$$

□

Problem 02

Maxwell's electromagnetism

Beginning with

$$\hat{F} := (F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad , \quad \hat{F}' := (F^{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

[1] Since naturally invariant under cyclical permutations, and transposition of two indices as well, as for example in swapping $\lambda \leftrightarrow \nu$:

$$\partial_\mu \underbrace{F_{\nu\lambda}}_{-F_{\lambda\nu}} + \underbrace{\partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu}}_{-\partial_\lambda F_{\nu\mu} - \partial_\nu F_{\mu\lambda}} = -\left(\partial_\mu F_{\lambda\nu} + \partial_\lambda F_{\nu\mu} + \partial_\nu F_{\mu\lambda}\right)$$

we write

$$T^{\mu\nu} = \underbrace{F^{\mu\lambda}}_{-F^{\lambda\mu}} \underbrace{F^{\nu}{}_{\lambda}}_{F^{\nu\varkappa}\eta_{\varkappa\lambda}} - \frac{1}{4}\eta^{\mu\nu} F^{\lambda\sigma} \underbrace{F_{\lambda\sigma}}_{-F_{\sigma\lambda}} = -\left(\hat{F}' \cdot \eta \cdot \hat{F}'\right)^{\nu\mu} + \frac{\eta^{\mu\nu}}{4} \underbrace{\text{trace}\left(\hat{F}' \cdot \hat{F}\right)}_{2(\mathbf{E}^2 - \mathbf{B}^2)}$$

$$= \begin{pmatrix} \mathbf{E}^2 & E_2B_3 - E_3B_2 & E_3B_1 - E_1B_3 & E_1B_2 - E_2B_1 \\ E_2B_3 - E_3B_2 & B_2^2 + B_3^2 - E_1^2 & -E_1E_2 - B_1B_2 & -E_1E_3 - B_1B_3 \\ E_3B_1 - E_1B_3 & -E_1E_2 - B_1B_2 & B_1^2 + B_3^2 - E_2^2 & -E_2E_3 - B_2B_3 \\ E_1B_2 - E_2B_1 & -E_1E_3 - B_1B_3 & -E_2E_3 - B_2B_3 & B_1^2 + B_2^2 - E_3^2 \end{pmatrix} + \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbf{E}^2 + \mathbf{B}^2)/2 & E_2B_3 - E_3B_2 & E_3B_1 - E_1B_3 & E_1B_2 - E_2B_1 \\ E_2B_3 - E_3B_2 & (\mathbf{E}^2 + \mathbf{B}^2)/2 - (E_1^2 + B_1^2) & -E_1E_2 - B_1B_2 & -E_1E_3 - B_1B_3 \\ E_3B_1 - E_1B_3 & -E_1E_2 - B_1B_2 & (\mathbf{E}^2 + \mathbf{B}^2)/2 - (E_2^2 + B_2^2) & -E_2E_3 - B_2B_3 \\ E_1B_2 - E_2B_1 & -E_1E_3 - B_1B_3 & -E_2E_3 - B_2B_3 & (\mathbf{E}^2 + \mathbf{B}^2)/2 - (E_3^2 + B_3^2) \end{pmatrix}$$

In particular:

$$T^{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \quad , \quad (T^{0i}) = (T^{i0}) = \mathbf{E} \times \mathbf{B}$$

$$(T^{ik}) = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \cdot 1_{3\times 3} - \mathbf{E} \otimes \mathbf{E} - \mathbf{B} \otimes \mathbf{B}$$

With

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu = -\frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + A_\mu J^\mu$$

$$= -\frac{1}{2}\Big[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\Big] + A_\mu J^\mu = -\frac{1}{2}\eta^{\mu\varkappa}\eta^{\nu\lambda}\Big[(\partial_\mu A_\nu)(\partial_\varkappa A_\lambda) - (\partial_\mu A_\nu)(\partial_\lambda A_\varkappa)\Big]$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} = -\frac{1}{2}\eta^{\mu\varkappa}\eta^{\nu\lambda} \left[\delta^\rho_\mu \delta^\sigma_\nu (\partial_\varkappa A_\lambda) + (\partial_\mu A_\nu)\delta^\rho_\varkappa \delta^\sigma_\lambda - \delta^\rho_\mu \delta^\sigma_\nu (\partial_\lambda A_\varkappa) - (\partial_\mu A_\nu)\delta^\rho_\lambda \delta^\sigma_\varkappa\right]$$

$$= -\eta^{\rho\varkappa}\eta^{\sigma\lambda}\left[\partial_\varkappa A_\lambda - \partial_\lambda A_\varkappa\right] = F^{\sigma\rho}$$

the Lagrange equations give

$$0 = \frac{\partial \mathcal{L}}{\partial A_\sigma} - \partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} = J^\sigma - \partial_\rho F^{\sigma\rho} \tag{0.5}$$

Thus, with $J^\mu = 0$ we may write

$$\partial_\nu T^{\mu\nu} = (\partial_\nu F^{\mu\lambda})F^\nu{}_\lambda + \underbrace{F^{\mu\lambda}\partial_\nu F^\nu{}_\lambda}_{F^\mu{}_\lambda \partial_\nu F^{\nu\lambda} \overset{(0.5)}{=} 0} - \frac{1}{4}\eta^{\mu\nu}(\partial_\nu F^{\lambda\sigma})F_{\lambda\sigma} - \frac{1}{4}\eta^{\mu\nu}\underbrace{F^{\lambda\sigma}(\partial_\nu F_{\lambda\sigma})}_{F_{\lambda\sigma}\partial_\nu F^{\lambda\sigma}}$$

$$= (\partial_\nu F^{\mu\lambda})F^\nu{}_\lambda - \frac{1}{2}\underbrace{\eta^{\mu\nu}F_{\lambda\sigma}\partial_\nu F^{\lambda\sigma}}_{F_{\lambda\sigma}\partial^\mu F^{\lambda\sigma}} = -\frac{1}{2}F_{\nu\lambda}\left[\partial^\mu F^{\nu\lambda} - \partial^\nu F^{\mu\lambda} - \partial^\nu F^{\mu\lambda}\right]$$

$$= -\frac{1}{2}F_{\nu\lambda}\underbrace{\left[\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu}\right]}_{=0 \text{ due to } (0.1)} = 0$$

which was to be proven.

Scalar field theory

Beginning with

$$T^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}(\partial_\lambda\phi)(\partial_\sigma\phi) - \frac{1}{2}\eta^{\mu\nu}\eta^{\lambda\sigma}(\partial_\lambda\phi)(\partial_\sigma\phi) - \eta^{\mu\nu}V(\phi)$$

$$= (\partial^\mu\phi)(\partial^\nu\phi) - \frac{\eta^{\mu\nu}}{2}\underbrace{(\partial^\sigma\phi)(\partial_\sigma\phi)}_{(\vec\nabla\phi)^2 - (\partial_0\phi)^2} - \eta^{\mu\nu}V$$

we see

$$T^{00} = \frac{1}{2}\left[(\vec\nabla\phi)^2 + \dot\phi^2\right] + V \quad , \quad (T^{i0}) = (T^{0i}) = \dot\phi\cdot(\vec\nabla\phi)$$

$$(T^{ij}) = (\vec\nabla\phi)\otimes(\vec\nabla\phi) - \frac{1}{2}\left[(\vec\nabla\phi)^2 - \dot\phi^2 + V\right]\cdot 1_{3\times3}$$

Furthermore, the Lagrange equation leads directly to

$$0 = \underbrace{\frac{\partial\mathcal{L}}{\partial\phi}}_{-\frac{\partial V}{\partial\phi}} - \partial_\mu\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}}_{-\partial^\mu\phi} = \underbrace{\partial_\mu\partial^\mu}_{\Box}\phi - \frac{\partial V}{\partial\phi} \tag{0.6}$$

Thus we may write

$$\partial_\nu T^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}(\partial_{\nu\lambda}\phi)(\partial_\sigma\phi) + \eta^{\mu\lambda}\eta^{\nu\sigma}(\partial_\lambda\phi)(\partial_{\nu\sigma}\phi) - \eta^{\mu\nu}\left[\frac{1}{2}\eta^{\lambda\sigma}(\partial_{\nu\lambda}\phi)(\partial_\sigma\phi) + \frac{1}{2}\eta^{\lambda\sigma}(\partial_\lambda\phi)(\partial_{\nu\sigma}\phi) + \partial_\nu V\right]$$

$$= (\partial^\mu\partial_\nu\phi)(\partial^\nu\phi)(\partial^\nu\phi) + (\partial^\mu\phi)\underbrace{(\partial^\nu\partial_\nu\phi)}_{\Box\phi} - \eta^{\mu\nu}\left[\eta^{\lambda\sigma}(\partial_{\nu\lambda}\phi)(\partial_\sigma\phi) + \underbrace{\partial_\nu\phi}_{\frac{\partial V}{\partial\phi}\partial_\nu\phi}\right]$$

$$= \cancel{(\partial^\mu\phi\partial_\nu\phi)(\partial^\nu\phi)} + (\partial^\mu\phi)\Box\phi - \cancel{(\partial^\mu\partial_\lambda\phi)(\partial^\lambda\phi)} - \frac{\partial V}{\partial\phi}(\partial^\mu\phi) = (\partial^\mu\phi)\underbrace{\left[\Box\phi - \frac{\partial V}{\partial\phi}\right]}_{\substack{0\\ \text{due to } (0.6)}} = 0$$

which was to be proven