General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 02 - Solutions

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Problem 01 (Carroll, Problem 1.3)

Let w.l.o.g. $c = 1$. Consider an inertial system O, and 2 boosts along the x-axis with two different velocities $v' = \tanh\varphi'$, $v'' = \tanh\varphi''$, so that the Lorentz-Transformations are given by

$$\Lambda' = \begin{pmatrix} \cosh\varphi' & -\sinh\varphi' & 0 & 0 \\ -\sinh\varphi' & \cosh\varphi' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \& \quad \Lambda'' = \begin{pmatrix} \cosh\varphi'' & -\sinh\varphi'' & 0 & 0 \\ -\sinh\varphi'' & \cosh\varphi'' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

respectively. As is known, the new x' & t' axes are within system O described by $t = x\tanh\varphi'$ & $t = x/\tanh\varphi'$ respectively, therefore oriented symmetrically along $t = x$ (similarly for t'' & x''). Figure (0.1) shows the 6 t, x, t', x', t'', x'' axes of the 3 coordinate systems for appropriate φ', φ''-values, along with 3 events ABC, observed in the order CBA, ABC & ACB in the systems O, O' & O'' respectively.

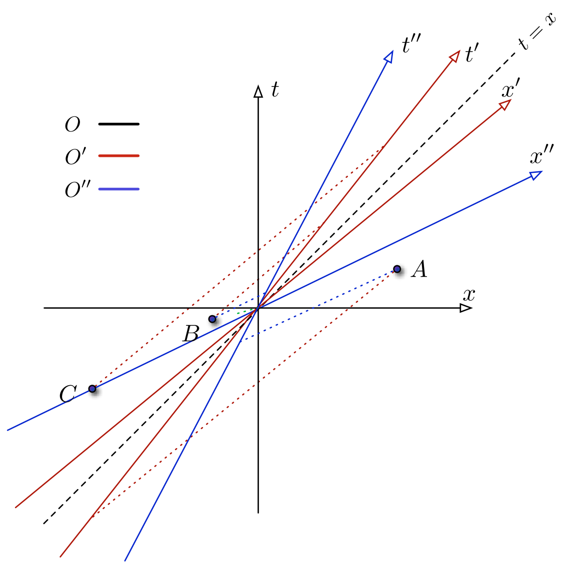

Figure 0.1: Space-time diagram on problem 01.

Problem 02 (Carroll, Problem 1.10)

The field-tensor $F_{\mu\nu}$ is given by

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

($-+++$ notation) transforming as

$$F_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}F_{\mu\nu} = \left[\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}'}\right)^T \cdot F \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{x}'}\right]_{\mu'\nu'}$$

(a) A rotation of the coordinate system about the y-axis by an angle ϑ is described by

$$\mathbf{x} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\vartheta & 0 & \cos\vartheta \end{pmatrix}}_{\Lambda} \cdot \mathbf{x}'$$

Thus

$$F_{\mu'\nu'} = \left(\Lambda^T F \Lambda\right)_{\mu'\nu'} = \begin{pmatrix} 0 & -E_1\cos\vartheta - E_3\sin\vartheta & -E_2 & E_1\sin\vartheta - E_3\cos\vartheta \\ E_1\cos\vartheta + E_3\sin\vartheta & 0 & B_3\cos\vartheta - B_1\sin\vartheta & -B_2 \\ E_2 & B_1\sin\vartheta - B_3\cos\vartheta & 0 & B_1\cos\vartheta + B_3\sin\vartheta \\ -E_1\sin\vartheta + E_3\cos\vartheta & B_2 & -B_1\cos\vartheta - B_3\sin\vartheta & 0 \end{pmatrix}$$

(b) A boost along the z-axis with velocity v is described by

$$\mathbf{x}' = \underbrace{\begin{pmatrix} \gamma & 0 & 0 & -v\gamma/c^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix}}_{\Lambda} \cdot \mathbf{x}$$

whereas

$$\gamma := \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Consequently, the field-tensor transforms as

$$F_{\mu'\nu'} = \left(\Lambda^T F \Lambda\right)_{\mu'\nu'} = \begin{pmatrix} 0 & -E_1\gamma - B_2 v\gamma & -E_2\gamma + B_1 v\gamma & E_3 v^2\gamma^2/c^2 - E_3\gamma^2 \\ E_1\gamma + B_2 v\gamma & 0 & B_3 & -E_1 v\gamma/c^2 - B_2\gamma \\ E_2\gamma - B_1 v\gamma & -B_3 & 0 & -E_2 v\gamma/c^2 + B_1\gamma \\ E_3\gamma^2 - v^2\gamma^2 E_3/c^2 & E_1 v\gamma/c^2 + B_2\gamma & E_2 v\gamma/c^2 - B_1\gamma & 0 \end{pmatrix}$$

Problem 03

(a) Due to isotropy of space, the movement of S' along x preserves the y & z coordinates, that is $y' = y,\ z' = z$. Thus we shall restrict our thoughts to $(t', x') = \mathbf{F}(t, x)$, whereas $\mathbf{F}(t, x) = \mathbf{F}(v^2; t, x)$. Since non-accelerated motions in S are also non-accelerated motions in S', the transformation $\mathbf{F}$ maps straight lines to straight lines and is thus affine. Matching the origins at $t = t' = 0$, reduces $\mathbf{F}$ to the linear form

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \underbrace{\begin{pmatrix} g & \beta \\ \gamma & f \end{pmatrix}}_{\hat{F}} \cdot \begin{pmatrix} t \\ x \end{pmatrix}$$

As the origin $x' = 0$ of S' is seen to be moving at constant speed v in S, it follows

$$\begin{pmatrix} t' \\ 0 \end{pmatrix} = \hat{F} \cdot \begin{pmatrix} t \\ vt \end{pmatrix} \tag{0.1}$$

and in particular $\gamma t + fvt = 0$, or $\gamma = -vf$:

$$\hat{F} = \begin{pmatrix} g & \beta \\ -vf & f \end{pmatrix}$$

Consider now the two events $(t = 0, x = 0)$ & $(t = T, x = 0)$ in S, appearing in S' as $(t' = 0, x' = 0)$ and $(t' = gT, x' = -vfT)$ respectively. Thus g can be interpreted as a factor of *time dilation* and should due to isotropy of space not depend on the signum of v, that is, $g(v) = g(-v)$ or $g = g(v^2)$.

Similarly, considering the two events $(t = 0, x = -L)$ & $(t = 0, x = L)$ in S, appearing in S' as $(t' = -\beta L, x' = -fL)$ & $(t' = \beta L, x' = fL)$ respectively, leads to an interpretation of f as a factor of *length dilation*, thus $f(v) = f(-v)$ or $f = f(v^2)$.

From Eq. (0.1) one also obtains $t' = (g + \beta v)t$ for the coordinates of the S'-origin. In analogy to above, it follows from the isotropy of space that $(g + \beta v)$ should not depend on the signum of v, thus $\beta(v) = -\beta(-v)$, or $\beta(v) = -vh(v^2)$ for some function h.

Thus, one obtains

$$\boxed{\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} g(v^2) & -vh(v^2) \\ -vf(v^2) & f(v^2) \end{pmatrix} \cdot \begin{pmatrix} t \\ x \end{pmatrix}}$$

(b) Consistency of the initial transformation and its inverse require $x = x(x'(x,t), t'(x,t))$, that is

$$x = f\overbrace{[fx - vft]}^{x'} + fv\overbrace{[gt - vhx]}^{t'} = \left[f^2 - v^2 fh\right] \cdot x + \left[fgv - vf^2\right] \cdot t$$

As the above relation must hold for all x, t for some constant v, f, g, h, one can conclude

$$f^2 - v^2 fh = 1 \quad \wedge \quad \underbrace{fg - f^2 = 0}_{\overset{f \neq 0}{\Longrightarrow} f = g} \tag{0.2}$$

Note: Up to now, the Galileian transformation is not excluded, and is obtained in the case $f = g = 1,\ h = 0$!

(c) W.l.o.g. $x' = u't'$. The speed of the entity is given in S by

$$u = \frac{dx(x', t')}{dt(x', t')} \overset{x' = x'(t')}{=} \frac{dx(x'(t'), t')}{dt'} \cdot \left(\frac{dt(x'(t'), t')}{dt'}\right)^{-1}$$

$$= \frac{d}{dt'}\left(fu't' + vft'\right)\left[\frac{d}{dt'}\left(gt' + vhu't'\right)\right]^{-1} \overset{f=g}{=} \frac{u' + v}{1 + vu'(h/f)}$$

Now let w.l.o.g. $v > 0$ and assume

$$u \geq u' \quad \forall\, u' \tag{0.3}$$

Thus, by the second postulate of relativity (universal limiting speed c) one expects $u \overset{u' \to c}{\longrightarrow} c$, thus

$$\lim_{u' \to c} u = \frac{c + v}{1 + vc(h/f)} = c \quad \Rightarrow \quad f = hc^2$$

On the other hand, if assumption (0.3) is false, that is, $u < u'$ for some u', then by continuity of $u(u')$ and the fact $u(u' = 0) \geq u'$, there would be a speed u'_0 for which $u(u'_0) = u'_0$. But this is a contradiction, as in the third inertial frame, moving at speed u'_0 with respect to both S and S', the later two would seem to be equal! Therefore:

$$\boxed{u = \frac{u' + v}{1 + vu'/c^2}}$$

Moreover, relations (0.2) imply

$$f = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

that is, the Lorentz-Transformation

$$\boxed{\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\ \\ \frac{t - vx/c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \end{pmatrix}}$$

□