General Theory of Relativity
FSU Jena - WS 2009/2010
Problem set 01 - Solutions

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Problem 01 (Carroll, Problem 1.4)

Let $\tau(t)$ be the time at which light emitted by the gas at time t reaches the observer, while at position $(x, y)(t)$. From Fig. (0.1) one can see

$$\tau(t) = \frac{D - x(t)}{c \cos \alpha(t)} + t \tag{0.1}$$

and

$$x(t) = vt \cos \vartheta \quad , \quad y(t) = vt \sin \vartheta \tag{0.2}$$

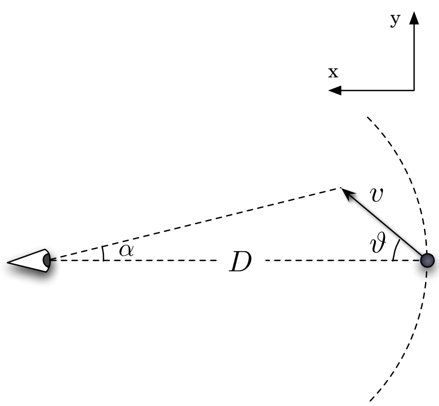

Figure 0.1: On Problem 01.

Thus the apparent transversal gas speed seems for the observer to be

$$v_{\text{app}} = \left.\frac{dy}{d\tau}\right|_{t=0} = \left.\frac{\dot{y}}{\dot{\tau}}\right|_{t=0} = \left.\frac{v \sin \vartheta}{\frac{-\dot{x}(t) \cos \alpha(t) + (D - x(t))\dot{\alpha}(t) \sin \alpha(t)}{c \cos^2 \alpha(t)} + 1}\right|_{t=0} \overset{\alpha(0)=0}{=} \frac{v \sin \vartheta}{1 - \frac{\dot{x}}{c}} = \frac{v \sin \vartheta}{1 - \frac{v}{c} \cos \vartheta}$$

Therefore, for

$$v > \frac{c}{\sin \vartheta + \cos \vartheta}$$

the apparent speed v_{app} exceeds c. An example would be $\vartheta = \frac{\pi}{4}$ with $v > c/\sqrt{2}$.

Problem 02 (Carroll, Problem 1.5)

With

$$\gamma := \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{E_{\mathrm{tot}}}{m_0 c^2}$$

the life-time of the particle appears for the experimentator to be bigger by the factor γ, that is, $\tau_{\mathrm{exp}} = 0.0207$ s. Furthermore, the muon's speed is given by

$$v = \sqrt{\frac{\gamma^2-1}{\gamma^2}} \cdot c = \sqrt{E^2 - m_0^2 c^4} \cdot \frac{c}{E} \quad ,$$

thus the muon would travel

$$v\tau_{\mathrm{exp}}/R \approx 12411 \text{ rad}$$

radii around the ring (radius R).

Problem 03 (Carroll, Problem 1.7)[1]

(a) $X^{\mu}{}_{\nu} \overset{\mathrm{def}}{=} X^{\mu\varkappa}\eta_{\varkappa\nu} = \begin{pmatrix} -6 & 0 & 1 & 0 \\ 1 & 0 & -12 & 2 \\ -1 & 6 & 0 & 0 \\ 8 & 1 & 1 & -6 \end{pmatrix}$

(b) $X_{\mu}{}^{\nu} \overset{\mathrm{def}}{=} \eta_{\mu\varkappa}X^{\varkappa\nu} = \begin{pmatrix} -6 & 0 & -1 & 0 \\ -1 & 0 & -12 & 2 \\ 1 & 6 & 0 & 0 \\ -8 & 1 & 1 & -6 \end{pmatrix}$

(c) $X^{(\mu\nu)} \overset{\mathrm{def}}{=} \frac{1}{2}\left(X^{\mu\nu} + X^{\nu\mu}\right) = \begin{pmatrix} 6 & -\frac{1}{2} & 1 & -4 \\ -\frac{1}{2} & 0 & -3 & \frac{3}{2} \\ 1 & -3 & 0 & \frac{1}{2} \\ -4 & \frac{3}{2} & \frac{1}{2} & -6 \end{pmatrix}$

(d) $X_{[\mu\nu]} \overset{\mathrm{def}}{=} \frac{1}{2}\left(X_{\mu\nu} - X_{\nu\mu}\right) = \frac{1}{2}\left(\eta_{\mu\rho}X^{\rho\sigma}\eta_{\sigma\nu} - \eta_{\nu\rho}X^{\rho\sigma}\eta_{\sigma\mu}\right) = \frac{1}{2}\begin{pmatrix} 6 & 0 & -1 & 0 \\ 1 & 0 & -12 & 2 \\ -1 & 6 & 0 & 0 \\ 8 & 1 & 1 & -6 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 6 & 1 & -1 & 8 \\ 0 & 0 & 6 & 1 \\ -1 & -12 & 0 & 1 \\ 0 & 2 & 0 & -6 \end{pmatrix}$

$= \begin{pmatrix} 0 & -\frac{1}{2} & 0 & -4 \\ \frac{1}{2} & 0 & -9 & \frac{1}{2} \\ 0 & 9 & 0 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

(e) $X^{\lambda}{}_{\lambda} = -12$

(f) $V^{\mu}V_{\mu} = -(-1)^2 + (-2)^2 + 0^2 + 2^2 = 7$

(g) $V_{\mu}X^{\mu\nu} = \left(-8, 2, 27, -16\right)$

[1] Different components