Redshifts of Photons

Stilianos Louca

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Abstract

Simple derivations are given for the cosmological & gravitational redshift as well as the special-relativistic Doppler effect, resulting from the proportionality of a photon's energy to the scalar product of its tangent vector and the observer's.

Contents

Conventions

- The 4-metric is always denoted by g.
- We shall denote 4-vectors by $\dot{x}$, with the actual point on the manifold implied by context.
- The spatial part of vectors shall be denoted by **bold** character, the time-part by t, that is $x = (t, \mathbf{x})$.
- Proper time & affine worldline parameters are symbolized by τ, derivatives with respect to τ by an upper dot. Derivatives with respect to the time coordinate t are denoted by primes $'$.
- The sub-index p will always denote *particles* and *photons* in particular, e and o emitter and observer respectively.
- Frequencies are symbolized by ν, angular frequencies by ω, wavelengths by λ.

1 Introduction

As is known from general relativity theory, a photon's energy as measured by some observer $\dot{x}_o$, is proportional to $g(\dot{x}_p, \dot{x}_o)$ with $x_p(\tau)$ being the geodesic photon worldline[5]. This is in direct analogy to the fact that, the observed energy of a massive particle of mass m the observed energy is given by $g(m\dot{x}_p, \dot{x}_o)$[4]. Note that a photon has no definite energy, as the latter need always be defined with respect to some observer. For a photon emitted by some emitter $\dot{x}_e$ and received by some observer $\dot{x}_o$, the respective frequency ratio is given by

$$\boxed{\frac{\nu_o}{\nu_e} = \frac{g(\dot{x}_p, \dot{x}_o)}{g(\dot{x}_p, \dot{x}_e)} \quad .} \tag{1.1}$$

This fundamental expression, will be used within the rest of this article to derive the cosmological and gravitational redshift, as well as the special-relativistic Doppler effect. While in principle frequency shifts are possible in both directions (i.e. *red*- and *blueshift*), we shall in the following for simplicity refer to both cases as *redshifts*.

Basic knowledge of special and general relativity theory is assumed. Otherwise, the reader is referred to Carroll[1, 2], Oloff[4] and Woodhouse[5].

2 The cosmological redshift

We shall illustrate the application of (1.1) to obtaining the cosmological redshift by assuming a metric of the form

$$g = c^2 \, dt^2 + R^2(t) \cdot g_s \tag{2.1}$$

in coordinates $(t, \mathbf{x})$, with the so called *scale factor* $R = R(t)$ and the 3-dimensional metric g_s defined on the sub-manifold $\{t : \text{const}\}$, thus acting (and depending) only on spatial parts[1]. The geodesic equation for the time-coordinate is given by

$$0 = \frac{d}{d\tau} \frac{\partial g(\dot{x}_p, \dot{x}_p)}{\partial \dot{t}_p} - \frac{\partial g(\dot{x}_p, \dot{x}_p)}{\partial t_p} = 2c^2 \, \ddot{t}_p - 2RR' \cdot g_s(\dot{\mathbf{x}}_p, \dot{\mathbf{x}}_p) \tag{2.3}$$

whereas for photons

$$0 = g(\dot{x}_p, \dot{x}_p) = c^2 \, \dot{t}_p^2 + R^2 \cdot g_s(\dot{\mathbf{x}}_p, \dot{\mathbf{x}}_p) \tag{2.4}$$

holds. Inserting (2.4) into (2.3) yields

$$0 = \ddot{t}_p + \frac{R'}{R} \cdot \dot{t}_p^2 = \ddot{t}_p + \frac{\dot{R}}{R} \cdot \dot{t}_p \tag{2.5}$$

[1] Assuming the cosmological principle[1] one obtains the known Robertson-Walker metric

$$g = c^2 \, dt^2 - R^2(t) \cdot \left[\frac{dr^2}{1 - \varepsilon r^2} + r^2 \, d\vartheta^2 + r^2 \sin^2 \vartheta \, d\varphi^2 \right] \tag{2.2}$$

in coordinates $(t, r, \vartheta, \varphi)$ with ε as constant sign of the space curvature.

which can be written as

$$\frac{\ddot{t}_p}{\dot{t}_p} = -\frac{\dot{R}}{R} \quad . \tag{2.6}$$

By integrating (2.6) on both sides one obtains

$$\dot{t}_p = \frac{C}{R(t_p)} \tag{2.7}$$

for some constant C. Now consider a photon x_p emitted by the emitter $\dot{x}_e$ and observed by observer $\dot{x}_o$, which we shall assume not to be moving, that is $\dot{\mathbf{x}}_e = 0 = \dot{\mathbf{x}}_o$. Then $\dot{t}_e = 1 = \dot{t}_o$ and applying (1.1) yields the cosmological redshift

$$\frac{\nu_o}{\nu_e} = \frac{c^2 \dot{t}_o \cdot \dot{t}_p\big|_o - R^2(t_o) \cdot g_s(\dot{\mathbf{x}}_p, \dot{\mathbf{x}}_o)}{c^2 \dot{t}_e \cdot \dot{t}_p\big|_e - R^2(t_e) \cdot g_s(\dot{\mathbf{x}}_p, \dot{\mathbf{x}}_e)} = \frac{\dot{t}_p\big|_o}{\dot{t}_p\big|_e} \quad , \tag{2.8}$$

which together with (2.7) takes the final form

$$\boxed{\frac{\nu_o}{\nu_e} = \frac{R(t_e)}{R(t_o)} \quad .} \tag{2.9}$$

This redshift is not to be confused with the special-relativistic Doppler effect (see later 3.2), as it is a pure effect of different scale factors (i.e. *universe sizes*) at emission and observation time.

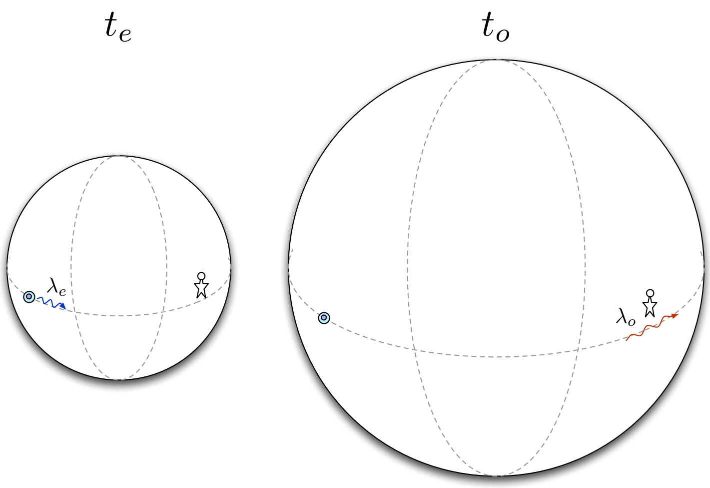

Figure 2.1: On the cosmological redshift. In an expanding universe, photons emitted by a resting source and observed by a resting observer, always appear redshifted.

3 Static metrics

We shall now specialize (1.1) to static metrics of the form

$$g = g_{tt}\, dt^2 + g_s \tag{3.1}$$

with the 3-dimensional metric g_s defined on $\{t : \text{const}\}$, acting and depending only on the spatial parts and $\partial_t g_{\mu\nu} = 0 \;\forall\; \mu, \nu$. By lemma A.1 the *energy*

$$E := g_{tt} \cdot \dot{t}_p \tag{3.2}$$

is conserved along photon geodesics. Thus for an emitter $\dot{x}_e$ and observer $\dot{x}_o$ the frequency ratio (1.1) takes the form

$$\frac{\nu_o}{\nu_e} = \frac{E \cdot \dot{t}_o + g_s(\dot{\mathbf{x}}_p, \dot{\mathbf{x}}_o)}{E \cdot \dot{t}_e + g_s(\dot{\mathbf{x}}_p, \dot{\mathbf{x}}_e)} \tag{3.3}$$

3.1 The gravitational redshift

Consider a photon x_p emitted by a resting emitter $\dot{x}_e$, observed by a resting observer $\dot{x}_o$. Then $g(\dot{x}_e, \dot{x}_e) = 1 = g(\dot{x}_o, \dot{x}_o)$ implies

$$\dot{t}_e = \left.\frac{1}{\sqrt{|g_{tt}|}}\right|_e \quad , \quad \dot{t}_o = \left.\frac{1}{\sqrt{|g_{tt}|}}\right|_o \tag{3.4}$$

and (3.3) yields the known gravitational redshift

$$\boxed{\frac{\nu_o}{\nu_e} = \sqrt{\frac{\left.|g_{tt}|\right|_e}{\left.|g_{tt}|\right|_o}} \quad .} \tag{3.5}$$

In case of a Schwarzschild metric

$$g = c^2 \left[1 - \frac{r_s}{r}\right] \, dt^2 - \left[1 - \frac{r_s}{r}\right]^{-1} dr^2 + r^2 \, d\vartheta^2 + r^2 \sin^2 \vartheta \, d\varphi^2 \tag{3.6}$$

with the Schwarzschild radius $r_s := \frac{2GM}{c^2}$, the ratio (3.5) takes the form

$$\boxed{\frac{\nu_o}{\nu_e} = \sqrt{\frac{1 - \frac{r_s}{r_e}}{1 - \frac{r_s}{r_o}}} \quad .} \tag{3.7}$$

Photons emitted from the surface of a massive star are thus redshifted for distant, resting observers.

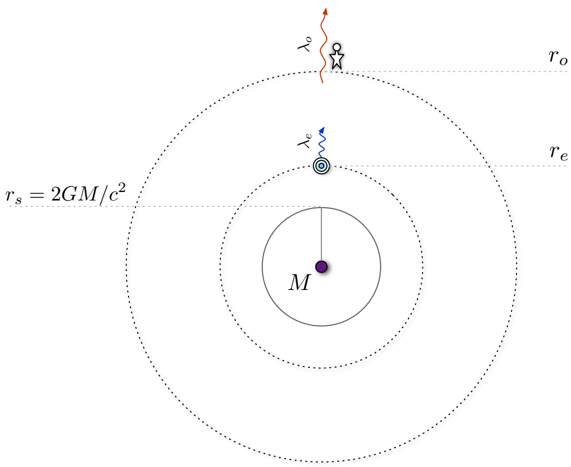

Figure 3.1: On the gravitational redshift in spherically symmetric, static metrics around massive stars. Photons emitted from nearby the star, appear redshifted to more distant observers.

If one defines the *redshift*

$$z := \frac{\lambda_o - \lambda_e}{\lambda_e} \quad , \tag{3.8}$$

then by using (3.7) in the case of a Schwarzschild metric for distances r_e, r_o much greater than the Schwarzschild radius[2] r_s, one obtains

$$z = \frac{\lambda_o}{\lambda_e} - 1 \approx \left[1 - \frac{r_s}{2r_o}\right] \left[1 + \frac{r_s}{2r_e}\right] - 1 \approx \frac{r_s}{2} \left[\frac{1}{r_e} - \frac{1}{r_o}\right] \quad . \tag{3.9}$$

[2]The Earths Schwarzschild radius is approximately 9 mm, the Suns about 3 km.

Interpreting $\Phi(r) := -\frac{GM}{r}$ as *gravitational potential* at distance r, allows one to write (3.9) as

$$\boxed{z \approx \frac{1}{c^2}\left[\Phi(r_o) - \Phi(r_e)\right] \quad .} \tag{3.10}$$

This expression for the redshift can, in the case of weak *gravitational fields*, also be derived[2] directly from the *Einstein equivalence principle*[3]. The case $|r_o - r_e| \ll r_e$ is illustrated in figure 3.2.

Figure 3.2: The gravitational redshift for radii much larger than the Schwarzschild radius, illustrated on the Earth's surface. The redshift is approximately given by $z \approx \frac{gh}{c^2}$ with $g \approx 9.8\ \mathrm{m} \cdot \mathrm{s}^{-2}$ as gravitational acceleration.

3.2 The special-relativistic Doppler effect

Consider the metric of flat spacetime

$$g = c^2\ dt^2 - d\mathbf{x}^2 \tag{3.11}$$

and a photon x_p emitted by the emitter $\dot{x}_e$, observed by $\dot{x}_o$. By lemma A.1 the component $\dot{t}_p$ is constant along the photon worldline and (1.1) takes the form

$$\frac{\nu_o}{\nu_e} = \frac{c^2\dot{t}_p \cdot \dot{t}_o - \dot{\mathbf{x}}_p\dot{\mathbf{x}}_o}{c^2\dot{t}_p \cdot \dot{t}_e - \dot{\mathbf{x}}_p\dot{\mathbf{x}}_e} = \frac{c^2\dot{t}_o - \frac{d\mathbf{x}_p}{dt_p} \cdot \dot{\mathbf{x}}_o}{c^2\dot{t}_e - \frac{d\mathbf{x}_p}{dt_p} \cdot \dot{\mathbf{x}}_e} = \frac{\dot{t}_o}{\dot{t}_e} \cdot \frac{c^2 - \frac{d\mathbf{x}_p}{dt_p}\frac{d\mathbf{x}_o}{dt_o}}{c^2 - \frac{d\mathbf{x}_p}{dt_p}\frac{d\mathbf{x}_e}{dt_e}} \quad . \tag{3.12}$$

Let $\mathbf{v}_o := \frac{d\mathbf{x}_o}{dt_o}$, $\mathbf{v}_e := \frac{d\mathbf{x}_e}{dt_e}$ and $\mathbf{e}_p := \frac{\dot{\mathbf{x}}_p}{\|\dot{\mathbf{x}}_p\|}$, then

$$\dot{t}_o = \frac{1}{\sqrt{1 - \frac{\mathbf{v}_o^2}{c^2}}} =: \gamma_o \quad , \quad \dot{t}_e = \frac{1}{\sqrt{1 - \frac{\mathbf{v}_e^2}{c^2}}} =: \gamma_e \tag{3.13}$$

and (3.13) can be written as

$$\boxed{\frac{\nu_o}{\nu_e} = \frac{\gamma_o}{\gamma_e} \cdot \frac{c - \mathbf{e}_p\mathbf{v}_o}{c - \mathbf{e}_p\mathbf{v}_e} \quad .} \tag{3.14}$$

The first factor in (3.14) can be attributed to time-dilational effects, with the second one merely corresponding to the non-relativistic Doppler effect. As an example, consider an emitter moving away from a resting observer at speed $\mathrm{v}_e := \|\mathbf{v}_e\|$, that is, $\mathbf{v}_e \parallel -\mathbf{e}_p$. Then (3.14) leads to the more familiar expression

$$\boxed{\frac{\nu_o}{\nu_e} = \sqrt{\frac{1 - \frac{\mathrm{v}_e}{c}}{1 + \frac{\mathrm{v}_e}{c}}}} \tag{3.15}$$

[3]Stating that the laws of physics reduce to those of special relativity in small enough regions of spacetime, thus making it impossible to detect a gravitational field.

of the special-relativistic Doppler effect. In the non-relativistic limit (3.14) simplifies to

$$z \approx \frac{\mathbf{e}_p}{c}(\mathbf{v}_o - \mathbf{v}_e) \quad , \tag{3.16}$$

with z as the redshift defined in (3.8). From (3.16) it follows readily that

$$\omega_o - \omega_e \approx \mathbf{k}_o \cdot (\mathbf{v}_e - \mathbf{v}_o) \approx \mathbf{k}_e \cdot (\mathbf{v}_e - \mathbf{v}_o) \quad . \tag{3.17}$$

A Appendix

A.1 Lemma: On the conservation of energy along geodesics

Let g be such that for some coordinate index $\varkappa$ all its components do not depend on $x^\varkappa$, that is

$$\partial_\varkappa g_{\mu\nu} = 0 \;\; \forall\; \mu, \nu \tag{A.1}$$

and g can be decomposed as

$$g = g_{\varkappa\varkappa} dx^\varkappa dx^\varkappa + g_s \quad , \tag{A.2}$$

with g_s as 3-dimensional metric, defined on the sub-manifold $\{x^\varkappa : \text{const}\}$. Then for any curve satisfying the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma \quad , \tag{A.3}$$

the value

$$E := \dot{x}^\varkappa \cdot g_{\varkappa\varkappa}$$

is constant along the curve.

Proof: Let $x = x(\tau)$. Then

$$\dot{E} = \ddot{x}^\varkappa g_{\varkappa\varkappa} + \dot{x}^\varkappa \dot{g}_{\varkappa\varkappa} \overset{(A.3)}{=} -\Gamma^\varkappa_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma g_{\varkappa\varkappa} + \dot{x}^\varkappa \dot{x}^\mu \partial_\mu g_{\varkappa\varkappa} = -\frac{g^{\varkappa\lambda}}{2} \left(\partial_\rho g_{\sigma\lambda} + \partial_\sigma g_{\rho\lambda} - \partial_\lambda g_{\rho\sigma} \right) \dot{x}^\rho \dot{x}^\sigma g_{\varkappa\varkappa} + \dot{x}^\varkappa \dot{x}^\mu \partial_\mu g_{\varkappa\varkappa}$$

$$= -\frac{g^{\varkappa\varkappa}}{2} (\partial_\rho g_{\sigma\varkappa} + \partial_\sigma g_{\rho\varkappa} - \underbrace{\partial_\varkappa g_{\rho\sigma}}_{0}) \dot{x}^\rho \dot{x}^\sigma g_{\varkappa\varkappa} + \dot{x}^\varkappa \dot{x}^\mu \partial_\mu g_{\varkappa\varkappa} = -\frac{g^{\varkappa\varkappa}}{2} \left(\dot{x}^\rho \dot{x}^\varkappa \partial_\rho g_{\varkappa\varkappa} + \dot{x}^\varkappa \dot{x}^\sigma \partial_\sigma g_{\varkappa\varkappa} \right) g_{\varkappa\varkappa} + \dot{x}^\varkappa \dot{x}^\mu \partial_\mu g_{\varkappa\varkappa}$$

$$= -\underbrace{g^{\varkappa\varkappa} g_{\varkappa\varkappa}}_{1} \dot{x}^\mu \dot{x}^\varkappa \partial_\mu g_{\varkappa\varkappa} 1 + \dot{x}^\mu \dot{x}^\varkappa \partial_\mu g_{\varkappa\varkappa} = 0$$

□

References

[1] S. M. Carroll, *Spacetime and Geometry - An Introduction to General Relativity*
Benjamin Cummings, 2003

[2] S. M. Carroll, *Lecture Notes on General Relativity*
University of Chicago, 1997
http://nedwww.ipac.caltech.edu/level5/March01/Carroll3/Carroll_contents.html (July 27, 2010)

[3] A. Liddle, *An Introduction to Modern Cosmology*
John Wiley & Sons, 2003

[4] R. Oloff, *Geometrie der Raumzeit: Eine mathematische Einführung in die Relativitätstheorie*
Vieweg & Teubner, 2010

[5] N. M. J. Woodhouse, *General Relativity*
Springer, 2007