Gewöhnliche Differentialgleichungen
FSU Jena - SS 2007
Serie 07 - Lösungen

Stilianos Louca

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Aufgabe 01

a)

$$2y^5y'' = -1 \overset{y_0\neq 0}{\to} y'' = -\frac{1}{2y^5} \ \to\ \frac{d}{dt}(y'^2) = 2y'y'' = -\frac{y'}{y^5} = \frac{d}{dt}\left(\frac{1}{4y^4}\right)$$

$$\Rightarrow\ y'^2 = \frac{1}{4y^4} + C,\ C : const \ \overset{AWP}{\rightsquigarrow}\ C = 0 \ \to\ y' = \frac{1}{2y^2} \ \to\ \frac{2}{3}y^3 = \int 2y^2 dy = \int dt = t + D,\ D : const$$

$$\overset{AWP}{\rightsquigarrow}\ D = -\frac{5}{3} \ \Rightarrow\ y = \sqrt[3]{\frac{3t}{2} - \frac{5}{2}}$$

b)

$$y'' = 2\sin 2y \ \to\ \frac{d}{dt}\left(y'^2\right) = 2y'y'' = 4y'\sin 2y = -\frac{d}{dt}(2\cos 2y)$$

$$\Rightarrow\ y'^2 = -2\cos 2y + C \ \overset{AWP}{\rightsquigarrow}\ C = 2 \ \to\ y' = \sqrt{2 - 2\cos 2y}$$

$$\Rightarrow\ t + D = \int dt = \int \frac{dy}{\sqrt{2}\cdot\sqrt{1-\cos 2y}} \overset{u:=\sqrt{1+\cos 2y}}{=} -\frac{1}{4}\cdot \ln\left|\frac{\sqrt{2}+\sqrt{1+\cos 2y}}{\sqrt{2}-\sqrt{1+\cos 2y}}\right| \ \overset{AWP}{\rightsquigarrow}\ D = 0$$

$$\to\ \cos 2y = 2\left(\frac{e^{-4t}-1}{e^{-4t}+1}\right)^2 - 1$$

c)

$$y'' = (y')^2 \sin y \overset{y'(0)\neq 0}{\to}\ Sub : p := y'(x(y)) \ \rightsquigarrow\ \frac{dp}{dy} = \frac{y''}{p} = \frac{p^2 \sin y}{p} = p\sin y$$

$$\Rightarrow\ \ln|p| = \int \frac{dp}{p} = \int \sin y\ dy = -\cos y + C \ \overset{AWP}{\rightsquigarrow}\ C = 1 \ \Rightarrow\ p = y' = e^{1-\cos y}$$

$$\Rightarrow\ \int_0^y e^{\cos u - 1} du = \int_0^t du = t$$

d)

$$3yy'y'' = (y')^3 - 1 \overset{y(0),y'(0)\neq 0}{\rightarrow} y'' = \frac{(y')^3-1}{3yy'},\ Sub: p := y'(y) \rightsquigarrow \frac{dp}{dy} = \frac{y''}{p} = \frac{p^3-1}{3yp^2}$$

$$\Rightarrow \ln\left|p^3-1\right| = \int \frac{3p^2\ dp}{p^3-1} = \int \frac{dy}{y} = \ln|y| + C \overset{AWP}{\rightsquigarrow} C = \ln 7 \Rightarrow y' = p = \sqrt[3]{7y+1}$$

$$\Rightarrow \frac{3(7y+1)^{2/3}}{14} - \frac{6}{7} = \frac{3}{14}\cdot\left[(7u+1)^{2/3}\right]_1^y = \int_1^y \frac{du}{\sqrt[3]{7u+1}} = \int_0^t dt = t \rightarrow y = \frac{1}{7}\cdot\left[\left(\frac{14t+12}{3}\right)^{3/2} - 1\right]$$

Aufgabe 02

a)

$$\dot{\vec{y}} = \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2y_1+y_2 \\ 3y_1+4y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}}_{A}\cdot\vec{y}$$

$$Triviale\ L\ddot{o}sung:\ \vec{y} \equiv 0$$

$$Ansatz:\ \vec{y} = e^{\lambda t}\cdot\vec{d} \rightarrow \dot{\vec{y}} = \lambda\vec{y} = A\vec{y} \rightarrow (A-\lambda E)\cdot\vec{d} = 0 \rightarrow \det(A-\lambda E) \overset{!}{=} 0$$

$$\rightsquigarrow \lambda_1 = 1,\ \lambda_2 = 5,\ (A-\lambda_i E)\cdot\vec{d_i} \overset{!}{=} 0 \rightsquigarrow \vec{d_1} = \alpha\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \vec{d_2} = \beta\cdot\begin{pmatrix} 1 \\ 3 \end{pmatrix},\ \alpha,\beta: const$$

$$\rightarrow \vec{y} = e^t\cdot\vec{d_1} + e^{5t}\cdot\vec{d_2}$$

b)

$$\dot{\vec{y}} = \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}}_{A}\cdot\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ Triviale:\ \vec{y} \equiv 0$$

$$Ansatz:\ \vec{y} = e^{\lambda t}\cdot\vec{d} \rightarrow \det(A-\lambda E) \overset{!}{=} 0 \rightsquigarrow \lambda = 2 \rightsquigarrow \vec{d} = \alpha\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Phi := \begin{pmatrix} 1 & e^{2t} \\ 0 & -e^{2t} \end{pmatrix},\ \Phi^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -e^{-2t} \end{pmatrix},\ \dot{\Phi} = \begin{pmatrix} 0 & 2e^{2t} \\ 0 & -2e^{2t} \end{pmatrix}$$

$$Ansatz:\ \vec{y} = \Phi\cdot\vec{X} \rightarrow \dot{\vec{y}} = \dot{\Phi}\vec{X} + \Phi\dot{\vec{X}} = A\vec{y} = A\Phi\vec{X} \rightarrow \dot{\vec{X}} = \underbrace{\Phi^{-1}(A\Phi - \dot{\Phi})}_{B}\vec{X}$$

$$B = \begin{pmatrix} 2 & 0 \\ -e^{-2t} & 0 \end{pmatrix} \rightarrow \dot{\vec{X}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = B\cdot\vec{X} = \begin{pmatrix} 2x_1 \\ -e^{-2t}x_1 \end{pmatrix}$$

$$\Rightarrow x_1 = \beta e^{2t},\ x_2 = -\beta t \rightarrow \vec{y} = \Phi\cdot\vec{X} = \beta\Phi\cdot\begin{pmatrix} e^{2t} \\ -t \end{pmatrix} = \beta e^{2t}\begin{pmatrix} 1-t \\ t \end{pmatrix} \Rightarrow \vec{y} = e^{2t}\cdot\begin{pmatrix} \alpha+\beta(1-t) \\ -\alpha+\beta t \end{pmatrix}$$

c)

$$\dot{\vec{y}} = \underbrace{\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}}_{A} \cdot \vec{y},\ Triviale:\ \vec{y} \equiv 0$$

$$Ansatz:\ \vec{y} = e^{\lambda t} \cdot \vec{d} \ \rightsquigarrow\ (A - \lambda E) \cdot \vec{d} \stackrel{!}{=} 0 \ \rightsquigarrow\ \lambda_{1,2} = 1 \pm 3i \ \rightsquigarrow\ \vec{d_1} = \alpha \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix},\ \vec{d_2} = \beta \cdot \begin{pmatrix} 1 \\ i \end{pmatrix},\ \alpha, \beta \in \mathbb{C}$$

$$\rightarrow\ \vec{y} = e^t \cdot \begin{pmatrix} \alpha e^{i3t} + \beta e^{-i3t} \\ -i\alpha e^{i3t} + i\beta e^{-i3t} \end{pmatrix}$$

d)

$$\dot{\vec{y}} = \underbrace{\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}}_{A} \cdot \vec{y},\ Triviale:\ \vec{y} \equiv 0$$

$$Ansatz:\ \vec{y} = e^{\lambda t} \cdot \vec{d} \ \rightsquigarrow\ (A - \lambda E) \cdot \vec{d} \stackrel{!}{=} 0 \ \rightsquigarrow\ \lambda_{1,2} = 2 \pm i \ \rightsquigarrow\ \vec{d_1} = \alpha \cdot \begin{pmatrix} 1 \\ 1+i \end{pmatrix},\ \vec{d_2} = \beta \cdot \begin{pmatrix} 1 \\ 1-i \end{pmatrix},\ \alpha, \beta \in \mathbb{C}$$

$$\rightarrow\ \vec{y} = e^{2t} \cdot \begin{pmatrix} \alpha e^{it} + \beta e^{-it} \\ \alpha(1+i)e^{it} + \beta(1-i)e^{-it} \end{pmatrix}$$

Aufgabe 03

a)

$$\dot{\vec{y}} = \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{A} \cdot \vec{y} + \underbrace{\begin{pmatrix} 2 \\ 4t \end{pmatrix}}_{\vec{g}},\ Homogene:\ \vec{y}_h = \alpha \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{d_1}} + \beta e^{2t} \cdot \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\vec{d_2}},\ \alpha, \beta \in \mathbb{R}$$

$$Ansatz:\ \vec{y}_p = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = u(t) \cdot \vec{d_1} + v(t) \cdot e^{2t} \cdot \vec{d_2} \ \rightarrow\ \dot{\vec{y}}_p = \dot{u} \cdot \vec{d_1} + \dot{v}e^{2t} \cdot \vec{d_2} + 2ve^{2t} \cdot \vec{d_2} \stackrel{!}{=} A \cdot \vec{y}_p + \vec{g}$$

$$\rightarrow\ \begin{pmatrix} \dot{u} + \dot{v}e^{2t} \\ \dot{u} - \dot{v}e^{2t} \end{pmatrix} = \begin{pmatrix} 2 \\ 4t \end{pmatrix} \ \rightarrow\ \dot{u} = 1 + 2t \ \wedge\ \dot{v} = (1-2t)e^{-2t}$$

$$\rightarrow\ u = \int (1+2t)dt = t + t^2,\ v = \int (1-2t)e^{-2t}dt = te^{-2t} \ \rightarrow\ \vec{y} = (\alpha + t + t^2) \cdot \vec{d_1} + \left(\beta e^{2t} + t\right) \cdot \vec{d_2}$$

b)

$$\vec{y} = \underbrace{\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}}_{A} \cdot \vec{y} + \underbrace{\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}}_{\vec{g}},\ Hom.:\ \lambda = 0 \ \rightsquigarrow\ \vec{y}_h = \alpha \cdot \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\vec{d_1}} + \beta \cdot \underbrace{\begin{pmatrix} 1+2t \\ -t \end{pmatrix}}_{\vec{d_2}}$$

$$Ansatz:\ \vec{y}_p = \begin{pmatrix} a\cos x + b\sin x \\ c\cos x + d\sin x \end{pmatrix} \ \rightsquigarrow\ \vec{y}_p = \begin{pmatrix} -2\cos x - 3\sin x \\ 2\sin x \end{pmatrix} \ \rightarrow\ y = \alpha \cdot \vec{d_1} + \beta \cdot \vec{d_2} + \begin{pmatrix} -2\cos x - 3\sin x \\ 2\sin x \end{pmatrix}$$

Aufgabe 04

a)

$$\dot{\vec{y}} = \underbrace{\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}}_{A} \cdot \vec{y} + \underbrace{\begin{pmatrix} e^{-x} \\ 1 \end{pmatrix}}_{\vec{g}},\ \vec{y}(0) = \begin{pmatrix} -\frac{5}{12} \\ \\ \frac{1}{20} \end{pmatrix},\ \vec{y}_h = \alpha e^t \cdot \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\vec{d}_1} + \beta e^{5t} \cdot \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\vec{d}_2}$$

$$Ansatz:\ \vec{y}_p = u(t) \cdot e^t \cdot \vec{d}_1 + v(t) \cdot e^{5t} \cdot \vec{d}_2 \ \rightsquigarrow\ \dot{u}e^t \cdot \vec{d}_1 + \dot{v}e^{5t} \cdot \vec{d}_2 = \begin{pmatrix} \dot{u}e^t + \dot{v}e^{5t} \\ \\ -\dot{u}e^t + 3\dot{v}e^{5t} \end{pmatrix} = \vec{g}$$

$$\rightsquigarrow\ \dot{u} = \frac{3e^{-2t}}{4} - \frac{e^{-t}}{4} \ \wedge\ \dot{v} = \frac{e^{-5t}}{4} + \frac{e^{-6t}}{4} \ \rightarrow\ u = -\frac{3e^{-2t}}{8} + \frac{e^{-t}}{4} \ \wedge\ v = -\frac{e^{-5t}}{20} - \frac{e^{-6t}}{24}$$

$$\Rightarrow\ \vec{y} = \left(\alpha e^t - \frac{3e^{-t}}{8} + \frac{1}{4}\right) \cdot \vec{d}_1 + \left(\beta e^{5t} - \frac{1}{20} - \frac{e^{-t}}{24}\right) \cdot \vec{d}_2$$

$$AWP \ \rightsquigarrow\ \alpha = \frac{31}{240} \ \wedge\ \beta = \frac{7}{80}$$

b)

$$\dot{\vec{y}} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \cdot \vec{y} + \underbrace{\begin{pmatrix} 2e^{2x} \sin x \\ 2e^{2x} \cos x \end{pmatrix}}_{\vec{g}}\ \vec{y}_h = \alpha e^{(2+i)x} \cdot \underbrace{\begin{pmatrix} 1 \\ 1+i \end{pmatrix}}_{\vec{d}_1} + \beta e^{(2-i)x} \cdot \underbrace{\begin{pmatrix} 1 \\ 1-i \end{pmatrix}}_{\vec{d}_2}$$

$$Ansatz:\ \vec{y}_p = u(x) \cdot e^{(2+i)x} \cdot \vec{d}_1 + v(x) \cdot e^{(2-i)x} \cdot \vec{d}_2 \ \rightsquigarrow\ \begin{pmatrix} \dot{u}e^{(2+i)x} + \dot{v}e^{(2-i)x} \\ \\ \dot{u}(1+i)e^{(2+i)x} + \dot{v}(1-i)e^{(2-i)x} \end{pmatrix} = 2e^{2x} \cdot \begin{pmatrix} \sin x \\ \\ \cos x \end{pmatrix}$$

$$\rightsquigarrow\ \dot{u} = i \sin x e^{-ix} - i = i \cos x \sin x + \sin^2 x - i \ \wedge\ \dot{v} = i - i \sin x e^{ix} = i - i \sin x \cos x + \sin^2 x$$

$$\rightsquigarrow\ u = \frac{i \sin^2 x}{2} - \frac{\sin 2x}{4} + \frac{x}{2} - ix \ \wedge\ v = \frac{i \cos^2 x}{2} - \frac{\sin 2x}{4} + \frac{x}{2} + ix$$

$$\rightarrow\ \vec{y} = e^{2x} \cdot \begin{pmatrix} e^{ix}(\alpha + u) + e^{-ix}(\beta + v) \\ \\ e^{ix}(1+i)(\alpha + u) + e^{-ix}(1-i)(\beta + v) \end{pmatrix},\ AWP \ \rightsquigarrow\ \alpha = -i,\ \beta = -\frac{3i}{2}$$

Aufgabe 05

$$\dot{\vec{y}} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \vec{y},\ \vec{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$Ansatz:\ \vec{y} = e^{\lambda x} \cdot \vec{d} \ \rightsquigarrow\ (A - \lambda E) \cdot \vec{d} = 0 \ \rightsquigarrow\ \lambda_1 = -1, \lambda_2 = -2, \lambda_3 = 2$$

$$\vec{d_1} = \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},\ \vec{d_2} = \beta \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\ \vec{d_3} = \gamma \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},\ \alpha, \beta\gamma \in \mathbb{R}$$

$$\rightarrow\ \vec{y} = \alpha e^{-x} \cdot \vec{d_1} + \beta e^{-2x} \cdot \vec{d_2} + \gamma e^{2x} \cdot \vec{d_3},\ AWP \ \rightsquigarrow\ \alpha = \frac{1}{3},\ \beta = \frac{1}{2},\ \gamma = \frac{1}{6}$$