# Gewöhnliche Differentialgleichungen
# FSU Jena - SS 2007
# Serie 06 - Lösungen

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### Aufgabe 01

a)

$$x^2y'' - 4xy' + 6y = 0 \quad Ansatz: \ y = z(t), \ t := \ln x \ \Rightarrow \ y' = z'(t) \cdot \frac{1}{x}, \ y'' = \frac{z''(t)}{x^2} - \frac{z'(t)}{x^2}$$

$$\rightsquigarrow \ z'' - 5z' + 6z = 0 \ \rightsquigarrow \ z = C_1e^{3t} + C_2e^{2t} \ \Rightarrow \ y = C_1x^3 + C_2x^2 \ , \ AWP \ \rightsquigarrow \ y = x^3 + x^2$$

b)

$$x^2y'' - 3xy' + 5y = 0 \quad Ansatz: \ y = z(t), \ t := \ln x \ \rightsquigarrow \ z'' - 4z' + 5z = 0$$

$$\rightsquigarrow \ z = e^{2t} \cdot (C_1 \cos t + C_2 \sin t) \ \rightarrow \ y = x^2 \cdot (C_1 \cos \ln x + C_2 \sin \ln x)$$

$$AWP: \rightsquigarrow \ y = x^2 \cdot (\cos \ln x - 2 \sin \ln x), \ x > 0$$

c)

$$x^2y'' - 3xy' + 5y = 0 \quad Ansatz: \ y_h = z(t), \ t := \ln x \ \rightsquigarrow \ z'' + 5z' + 6z = 0$$

$$\rightsquigarrow \ z = C_1e^{-3t} + C_2e^{-2t} \ \rightarrow \ y_h = C_1x^{-3} + C_2x^{-2}$$

$$Ansatz: \ y_p = \frac{u(x)}{x^3} + \frac{v(x)}{x^2}, \ \frac{u'}{x^3} + \frac{v'}{x^2} = 0 \ \wedge \ \frac{3u'}{x^2} + \frac{2v'}{x} = -e^x \ \rightsquigarrow \ v' = xe^x \ \wedge \ u' = -x^2e^x$$

$$\rightarrow \ u = -e^x(x^2 - 2x + 2) \ \wedge \ v = e^x(x-1) \ \rightarrow \ y = y_h + e^x\frac{(x-2)}{x^3}$$

$$AWP: \rightsquigarrow \ y = \frac{e}{x^3} - \frac{1}{x^2} + e^x\frac{(x-2)}{x^3}$$

d)

$$x^2y'' - xy' + y = 8x^3 \quad Ansatz: \ y = z(t) \ t := \ln x \ \rightsquigarrow \ z = e^t(C_1 + C_2t) + 2e^{3t} \ \rightarrow \ y = x(C_1 + C_2 \ln x) + 2x^3$$

$$AWP: \rightsquigarrow \ y = -x(2 + 4\ln x) + 2x^3$$

## Aufgabe 02

a)

$$x(x^2+6)y'' - 4(x^2+3)y' + 6xy = 0,\ y_1(x) = x^3$$

$$Ansatz:\ y = y_1(x)\cdot\int u(x)dx \ \rightarrow\ y' = 3x^2\int udx + x^3u \ \wedge\ y'' = 6x\int udx + 6x^2u + x^3u'$$

$$\rightsquigarrow\ u'(6x+x^3) + u(24+2x^2) = 0 \ \rightsquigarrow\ u = \frac{(x^2+6)}{x^4}\cdot C,\ \Rightarrow\ y = C_1\cdot y_1 - C_2\cdot y_1\cdot\int udx = C_1\cdot x^3 + C_2\cdot(x^2+2)$$

$$AWP: \rightsquigarrow\ y = x^3 + 2(x^2+2)$$

b)

$$x(2x+1)y'' + 2(x+1)y' - 2y = 0,\ y_1(x) = \frac{1}{x}\ ,\ Ansatz:\ y = y_1\cdot\int udx \ \rightsquigarrow\ u'(2x+1) - 2u = 0 \ \rightsquigarrow\ u = (2x+1)$$

$$y = C_1y_1 + C_2y_1\cdot\int udx = \frac{C_1}{x} + C_2(x+1),\ AWP: \rightsquigarrow\ y = -\frac{1}{x} + 2(x+1)$$

c)

$$x^3y''' + (3x^3-6x^2)y'' + (2x^3-12x^2+18x)y' - (4x^2-18x+24)y = 0,\ y_1(x) = x^2$$

$$Ansatz:\ y = y_1\cdot\int udx = x^2\int udx,\ y' = 2x\int udx + x^2u,\ y'' = 2\int udx + 4xu + x^2u',\ y''' = 6u + 6xu' + x^2u''$$

$$\rightarrow\ x^5u'' + 3x^5u' + 2ux^5 = 0 \ \rightarrow\ u'' + 3u' + 2u = 0 \ \rightsquigarrow\ u_1 = e^{-x},\ u_2 = e^{-2x}$$

$$\Rightarrow\ y = C_1y_1 - C_2y_1\cdot\int u_1dx - 2C_3y_1\cdot\int u_2dx = x^2\cdot\left(C_1 + C_2e^{-x} + C_3e^{-2x}\right)$$

d)

$$xy'' - (2x+1)y' + 2y = 0,\ y_1 = e^{2x},\ Ansatz:\ y = y_1\int udx \ \rightsquigarrow\ u'x + u(2x-1) = 0 \ \rightsquigarrow\ u = xe^{-2x}$$

$$\Rightarrow\ y = C_1y_1 - 4C_2y_1\cdot\int udx = C_1e^{2x} + C_2\cdot(2x+1)$$

## Aufgabe 03

$$(1+x^2)\cdot y'' + 2xy' - 2y = 0 \ Ansatz: \ y = \sum_{k=0}^{\infty} a_k x^k \ \rightarrow \ y' = \sum_{k=1}^{\infty} a_k k x^{k-1}, \ y'' = \sum_{k=2}^{\infty} a_k k(k-1)x^{k-2}$$

$$\Rightarrow \ \sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k + \sum_{k=2}^{\infty} a_k(k-1)kx^k + 2\cdot\sum_{k=1}^{\infty} a_k k x^k - 2\cdot\sum_{k=0}^{\infty} a_k x^k$$

$$= 2(a_2 - a_0) + 6a_3 x + \sum_{k=2}^{\infty}\{a_{k+2}(k+1)(k+2) + a_k(k-1)k + 2a_k k - 2a_k\}\cdot x^k = 0$$

$$\Rightarrow \ a_2 = a_0, \ a_1 \ frei, \ a_3 = 0, \ a_{k+2} = -\frac{k(k-1)+2(k-1)}{(k+1)(k+2)}\cdot a_k = -\frac{(k-1)}{(k+1)}\cdot a_k$$

$$\rightarrow \ Für \ k = 2n+1, \ n \in \mathbb{N} : a_k = 0.$$

$$Für \ k = 2n : Induktionsannahme: \ a_k = \frac{(-1)^{\frac{k}{2}+1}a_0}{k-1} =, \ k = 0,2,4,6...$$

$$Induktionsanfang: \ a_0 = a_0 \ Klar$$

$$Induktionsschritt: \ a_{k+2} = -\frac{k-1}{k+1}\cdot a_k = -\frac{(-1)^{\frac{k}{2}+1}a_0}{k+1} = \frac{(-1)^{\frac{k+2}{2}+1}a_0}{(k+2)-1}$$

$$\Rightarrow \ y = a_1 + a_0\cdot\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cdot x^{2n}$$

## Aufgabe 04

$$(1-x^2)y'' - xy' + y = 0, \ y_1 = x \ Ansatz: \ y = y_1\cdot\int u dx \ \rightsquigarrow \ u'x(1-x^2) + u(2-3x^2) = 0$$

$$\rightsquigarrow \ \ln|u| = \int \frac{(3x^2-2)}{x(1-x^2)}\,dx = \int\left\{\frac{-2}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right\}dx = -2\ln|x| - \frac{\ln|1-x|}{2} - \frac{\ln|1+x|}{2} + C$$

$$\rightarrow \ u = \frac{1}{x^2\sqrt{1-x^2}} \ \rightarrow \ y = C_1 y_1 - C_2 y_1\cdot\int u dx = C_1 x + C_2\sqrt{1-x^2}$$

## Aufgabe 05

$$(x-1)y''-xy'+y=0 \quad Ansatz:\ y=\sum_{k=0}^{\infty}a_kx^k \ \rightarrow\ y'=\sum_{k=1}^{\infty}a_kkx^{k-1},\ y''=\sum_{k=2}^{\infty}a_kk(k-1)x^{k-2}$$

$$\rightarrow\ \sum_{k=1}k(k+1)a_{k+1}x^k-\sum_{k=0}^{\infty}a_{k+2}(k+1)(k+2)x^k-\sum_{k=1}^{\infty}ka_kx^k+\sum_{k=0}^{\infty}a_kx^k$$

$$=(a_0-2a_2)+x(2a_2-6a_3)+\sum_{k=2}^{\infty}\{-a_{k+2}(k+1)(k+2)+a_{k+1}k(k+1)+a_k(1-k)\}\cdot x^k=0$$

$$\rightarrow\ a_0=2a_2=6a_3,\ a_1:frei,\ a_{k+2}=\frac{k(k+1)a_{k+1}+(1-k)a_k}{(k+1)(k+2)},\ k\geq 2$$

$$Induktionsannahme:\ a_k=\frac{a_0}{k!},\ k=2,3,4...$$

$$Induktionsanfang:\ a_2=\frac{a_0}{2},\ a_3=\frac{a_0}{6}$$

$$Induktionsschritt:\ a_{k+1}=\frac{a_0}{k(k+1)}\cdot\left(\frac{(k-1)k}{k!}-\frac{(k-2)}{(k-1)!}\right)=\frac{a_0[(k-1)-(k-2)]}{k(k+1)(k-1)!}=\frac{a_0}{(k+1)!}$$

$$\Rightarrow\ y=a_0+a_1x+a_0\cdot\sum_{k=2}^{\infty}\frac{x^k}{k!}=\left(a_1-\frac{a_0}{2}\right)\cdot x+a_0e^x,\ AWP:\rightsquigarrow\ y=e^x$$

## Aufgabe 06

a)

$$y''+y'+\lambda y=0$$

$$Fall\ 1:\ \lambda<\frac{1}{4}\ \rightsquigarrow\ y=e^{-\frac{x}{2}}\cdot\left(C_1e^{x\omega}+C_2e^{-x\omega}\right),\ \omega=\frac{1}{2}\cdot\sqrt{1-4\lambda}$$

$$NB:\rightsquigarrow\ C_2=-C1,\ C_1e^{\omega}(1-2\omega)=C_1e^{-\omega}(1+2\omega)\ \rightarrow\ C_1=C_2=0$$

$$Fall\ 2:\ \lambda=\frac{1}{4}\ \rightsquigarrow\ y=e^{-\frac{x}{2}}\cdot(C_1+xC_2),\ NB:\rightsquigarrow\ C_1=C_2=0$$

$$Fall\ 3:\ \lambda>\frac{1}{4}\ \rightsquigarrow\ y=e^{-\frac{x}{2}}\cdot(C_1\cos\omega x+C_2\sin\omega x),\ \omega=\sqrt{\lambda-\frac{1}{4}}$$

$$NB:\rightsquigarrow\ C_1=0,C_2\neq 0,\ \omega=k\pi,\ k\in\mathbb{N}\ \rightarrow\ \lambda=(k\pi)^2+\frac{1}{4},\ y=C_2e^{-\frac{x}{2}}\sin(k\pi x)$$

b)

$$y'' + 2y' + (1-\lambda)y = 0$$

$$Fall\ 1:\ \lambda > 0 \ \rightsquigarrow\ y = e^{-x} \cdot \left(C_1 e^{x\sqrt{\lambda}} + C_2 e^{-x\sqrt{\lambda}}\right),\ RB: \rightsquigarrow\ C_1 e^{\sqrt{\lambda}}(1-\sqrt{\lambda}) = C_1 e^{-\sqrt{\lambda}}(1+\sqrt{\lambda}) \ \rightarrow\ C_1 = C_2 = 0$$

$$Fall\ 2:\ \lambda = 0 \ \rightsquigarrow\ y = e^{-x} \cdot (C_1 + xC_2),\ RB: \rightsquigarrow\ C_1 = 0 \ \wedge\ C_2 \in \mathbb{R} \setminus \{0\} \ \rightsquigarrow\ y = C_2 x e^{-x}$$

$$Fall\ 3:\ \lambda < 0 \ \rightsquigarrow\ y = e^{-x} \cdot (C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x) \ \rightsquigarrow\ C_1 = 0,\ C_2\sqrt{\lambda}\cos\sqrt{\lambda} = C_2 \sin\sqrt{\lambda} \ \rightsquigarrow\ \sqrt{\lambda} = \tan\sqrt{\lambda}$$

$$\rightsquigarrow\ y = e^{-x} \cdot C_2 \sin\left(x\sqrt{\lambda}\right),\ C_2 \neq 0$$

## Aufgabe 07

$$(xy')' + \frac{\lambda}{x}y = xy'' + y' + \frac{\lambda}{x}y = 0 \ \rightarrow\ y''x^2 + y'x + \lambda y = 0 \ \ Ansatz:\ y = z(t),\ t = \ln x \ \rightsquigarrow\ z'' + \lambda z = 0$$

$$Fall\ 1:\ \lambda < 0 \ \rightsquigarrow\ z = C_1 e^{t\omega} + C_2 e^{-t\omega},\ \omega = \sqrt{-\lambda} \ \rightarrow\ y = C_1 x^{\omega} + C_2 x^{-\omega},\ NB: \rightsquigarrow\ C_1 = C_2 = 0$$

$$Fall\ 2:\ \lambda = 0 \ \rightsquigarrow\ z = C_1 t + C_2 \ \rightarrow\ y = C_1 \ln x + C_2 \ NB: \rightsquigarrow\ C_1 = C_2 = 0$$

$$Fall\ 3:\ \lambda > 0 \ \rightsquigarrow\ z = C_1 \cos\omega t + C_2 \sin\omega t,\ \omega = \sqrt{\lambda} \ \rightarrow\ y = C_1 \cos(\omega \ln x) + C_2 \sin(\omega \ln x)$$

$$NB: \rightsquigarrow\ C_1 = 0,\ C_2 \neq 0 : \cdot \sin\omega = 0 \ \Rightarrow\ \omega = \sqrt{\lambda} = k\pi,\ k \in \mathbb{N} \rightarrow\ y = C_2 \sin(k\pi \ln x)$$