Gewöhnliche Differentialgleichungen
FSU Jena - SS 2007
Serie 04 - Lösungen

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28. Juli 2007

Aufgabe 01

Im allgemeinen seien $U \cong U(x,y),\ P \cong P(x,y) = \partial_x U,\ Q \cong Q(x,y) = \partial_y U$

a)

$$P(x,y) + Q(x,y)y' = 0,\ P(x,y) := (5x-2y),\ Q(x,y) := (y-2x),\ \partial_y P = -2 = \partial_x Q \ \Rightarrow\ Exakt.$$

$$U = \int P dx = \int (5x-2y)dx = \frac{5x^2}{2} - 2xy + C(y) \ \Rightarrow\ \partial_y U = -2y + C'(y) \stackrel{!}{=} Q(x,y) = y - 2y \ \Rightarrow\ C(y) = \frac{y^2}{2}$$

$$\Rightarrow\ U = \frac{5x^2}{2} - 2xy + \frac{y^2}{2} = const,\quad AWP\ \rightsquigarrow\ U = 1$$

b)

$$P + Qy' = 0,\ P = \frac{1}{y} + x,\ Q = -\frac{x}{y^2},\ \partial_y P = -\frac{1}{y^2} = \partial_x Q \ \Rightarrow\ Exakt$$

$$U = \int \left(\frac{1}{y} + x\right) dx = \frac{x}{y} + \frac{x^2}{2} + C(y) \ \Rightarrow\ \partial_y U = -\frac{x}{y^2} + C'(y) \stackrel{!}{=} -\frac{x}{y^2} \ \Rightarrow\ C(y) = const$$

$$\Rightarrow\ U = \frac{x}{y} + \frac{x^2}{2} = const,\quad AWP\ \rightsquigarrow\ U = 1$$

c)

$$P + Qy' = 0,\ P = 2x - 9x^2y^2,\ Q = 4y^3 - 6x^3y,\ \partial_y P = -18x^2y = \partial_x Q \ \Rightarrow\ Exakt$$

$$U = \int P dx = x^2 - 3x^3y^2 + C(y) \ \Rightarrow\ \partial_y U = -6x^3y^2 + C'(y) \stackrel{!}{=} 4y^3 - 6x^3y \ \Rightarrow\ C(y) = \int 4y^3 dy = y^4$$

$$\Rightarrow\ U = x^2 - 3x^3y^2 + y^4 = const,\quad AWP\ \rightsquigarrow\ U = 1$$

d)

$$P + Qy' = 0,\ P = 2xe^y - 1,\ Q = x^2e^y + 1,\ \partial_y P = 2xe^y = \partial_x Q \ \Rightarrow\ Exakt$$

$$U = \int P dx = x^2e^y - x + C(y) \ \Rightarrow\ \partial_y U = x^2e^y + C'(y) \stackrel{!}{=} x^2e^y + 1 \ \Rightarrow\ C(y) = \int dy = y$$

$$\Rightarrow\ U = x^2e^y - x + y = const,\quad AWP\ \rightsquigarrow\ U = 0$$

e)

$$P + Qy' = 0,\ P = \cos y + 2xy,\ Q = x^2 - y - x\sin y,\ \partial_y P = -\sin y + 2x = \partial_x Q \ \Rightarrow\ Exakt$$

$$U = \int P dx = x\cos y + x^2 y + C(y) \ \Rightarrow\ \partial_y U = -x\sin y + x^2 + C'(y) \overset{!}{=} x^2 - y - x\sin y \ \Rightarrow\ C(y) = \int -y dy = -\frac{y^2}{2}$$

$$\Rightarrow\ U = x\cos y + x^2 y - \frac{y^2}{2} = const,\quad AWP \ \rightsquigarrow\ U = -1$$

Aufgabe 02

Behandeln DGL der Form: $P + Qy' = 0$. M sei dabei jeweils der gesuchte Integrierende Faktor.

a)

$$P = 4x + 3y^2,\ Q = 2xy,\ \partial_y P = 6y \neq \partial_x Q = 2y \ \Rightarrow\ Nicht\ Exakt$$

$$\frac{\partial_y P - \partial_x Q}{Q} = \frac{2}{x} =: f(x) \ \rightarrow\ M := e^{\int \frac{2}{x} dx} = x^2$$

$$\Rightarrow\ U = \int MP dx = x^4 + x^3 y^2 + C(y) \ \Rightarrow\ \partial_y U = 2x^3 y + C'(y) \overset{!}{=} MQ = 2x^3 y \ \Rightarrow\ C = const$$

$$\Rightarrow\ U = x^4 + x^3 y^2 = const$$

b)

$$P = -2xy,\ Q = 3x^2 - y^2,\ \partial_y P = -2x \neq \partial_x Q = 6x \ \Rightarrow\ Nicht\ Exakt$$

$$\frac{\partial_y P - \partial_x Q}{P} = \frac{4}{y} \ \Rightarrow\ M := e^{-\int \frac{4}{y}} = \frac{1}{y^4}$$

$$U = \int MP dx = -\frac{x^2}{y^3} + C(y) \ \Rightarrow\ \partial_y U = \frac{3x^2}{y^4} + C'(y) \overset{!}{=} MP = \frac{3x^2}{y^4} - \frac{1}{y^2} \ \Rightarrow\ C = \frac{1}{y}$$

$$\Rightarrow\ U = -\frac{x^2}{y^3} + \frac{1}{y}$$

c)

$$P = \sin x - x\cos x - 3x^2(y-x)^2,\ Q = 3x^2(y-x)^2,\ \partial_y P = -6x^2(y-x) \neq 6x(y-x)(y-2x) = \partial_x Q \ \Rightarrow\ Nicht\ Exakt$$

$$\frac{\partial_y P - \partial_x Q}{Q} = -\frac{2}{x} \ \Rightarrow\ M := e^{-\int \frac{2}{x}} = \frac{1}{x^2}$$

$$U = \int MP dx = -\frac{\sin x}{x} + (y-x)^3 + C(y) \ \Rightarrow\ \partial_y U = 3(y-x)^2 + c'(y) \overset{!}{=} MQ = 3(y-x)^2 \ \Rightarrow\ C = const$$

$$\Rightarrow\ U = -\frac{\sin x}{x} + (y-x)^3 = const,\ x \neq 0$$

d)

$$P = x + y,\ Q = -\frac{x^2}{y},\ \partial_y P = 1 \neq \partial_x Q = -2\frac{x}{y}\ \Rightarrow\ Nicht\ Exakt$$

$$\frac{\partial_y P - \partial_x Q}{xP - yQ} = \frac{1}{xy}\ \rightarrow\ M = e^{\int \frac{1}{xy} d(xy)} = \frac{1}{xy}$$

$$U = \int MP dx = \frac{x}{y} + \ln|x| + C(y)\ \Rightarrow\ \partial_y U = -\frac{x}{y^2} + C'(y) \overset{!}{=} MQ = -\frac{x}{y^2}\ \Rightarrow\ C(y) = const$$

$$\Rightarrow\ U = \frac{x}{y} + \ln|x| = const,\ x, y \neq 0,\ AWP\ \rightsquigarrow\ U = -1\ \Rightarrow\ y = \frac{x}{-1 - \ln|x|},\ x \neq \pm\frac{1}{e}$$

Aufgabe 03

Zeigen: $\partial_y MP = \partial_x MQ$.

$$\partial_x(MQ) = Q\partial_x M + M\partial_x Q = Qe^{\int f(x)dx} \cdot f(x) + e^{\int f(x)dx}\partial_x Q = e^{\int f(x)dx} \cdot [\partial_y P - \partial_x Q + \partial_x Q] = e^{\int f(x)dx} \cdot \partial_y P = \partial_y(MP) \quad \square$$

Aufgabe 04

Bezeichne: $y_f(x) := y(x; f),\ y_g(x) := y(x; g)$

$$\sigma + L\,|y_f(x) - y_g(x)| = \sigma + L \cdot \left| \int_{x_0}^{x} f(t, y_f(t))dt - \int_{x_0}^{x} g(t, y_g(t))dt \right|$$

$$= \sigma + L \cdot \left| \int_{x_0}^{x} f(t, y_f(t))dt - \int_{x_0}^{x} f(t, y_g(t))dt + \int_{x_0}^{x} f(t, y_g(t))dt - \int_{x_0}^{x} g(t, y_g(t))dt \right|$$

$$\leq \sigma + L \cdot \left| \int_{x_0}^{x} \{|f(t, y_f(t)) - f(t, y_g(t))| + |f(t, y_g(t)) - g(t, y_g(t))|\}\, dt \right| \leq \sigma + L \cdot \left| \int_{x_0}^{x} \{L \cdot |y_f(t) - y_g(t)| + \sigma\}\, dt \right|$$

$$\Rightarrow\ \sigma + L\,|y_f(x) - y_g(x)| \leq \sigma \cdot e^{L \cdot |x - x_0|}\ \Rightarrow\ |y_f(x) - y_g(x)| \leq \frac{\sigma}{L} \cdot \left(e^{L \cdot |x - x_0|} - 1\right) \quad \square$$

Aufgabe 05

Wir setzen $y_1 := y,\ y_1' =: y_2,\ y_2' = -y_1$, definieren das Folgenpaar

$$y_1^{n+1} := 0 + \int_0^x y_2^n(x)dx,\ y_2^{n+1} := 1 + \int_0^x -y_1^n(x)dx,\ y_1^0 = 0,\ y_2^0 = 1$$

und zeigen dass für $n = 1, 3, 5...$

$$y_1^n = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} ... \frac{x^n}{n!}\ \wedge\ y_2^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} ... \frac{x^{n-1}}{(n-1)!}$$

bzw. für $n = 2, 4, 6...$

$$y_1^n = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} ... \frac{x^{n-1}}{(n-1)!}\ \wedge\ y_2^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} ... \frac{x^n}{n!}$$

gilt.
Induktionsanfang:

$$y_1^1 = 0 + \int_0^x y_2^0(x)dx = \int_0^x dx = x, \quad y_2^1 = 1 + \int_0^x -y_1^0(x)dx = 1 + \int_0^x 0dx = 1$$

$$y_1^2 = \int_0^x y_2^1(x)dx = \int_0^x dx = x, \quad y_2 = 1 + \int_0^x -y_1^1(x)dx = 1 + \int_0^x -x\ dx = 1 - \frac{x^2}{2}$$

Induktionsannahme: Siehe oberes Beweisziel.
Induktionsschritt: Für $n = 1, 3, 5...$ also $n + 1 = 2, 4, 6...$

$$y_1^{n+1} = \int_0^x y_2^n = \int_0^x \left\{ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} ... \frac{x^{n-1}}{(n+1)!} \right\} dx = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} ... + \frac{x^n}{n!}$$

$$y_2^{n+1} = 1 - \int_0^x y_1^n = 1 - \int_0^x \left\{ \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} ... \frac{x^n}{n!} \right\} dx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} ... \frac{x^{n+1}}{(n+1)!}$$

Für $n = 2, 4, 6...$ also $n + 1 = 3, 5, 7...$

$$y_1^{n+1} = \int_0^x y_2^n = \int_0^x \left\{ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} ... \frac{x^n}{n!} \right\} dx = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} ... \frac{x^{n+1}}{(n+1)!}$$

$$y_2^{n+1} = 1 - \int_0^x y_1^n = 1 - \int_0^x \left\{ \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} ... \frac{x^{n-1}}{(n-1)!} \right\} dx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} ... \frac{x^n}{n!}$$

Also

$$y_1 \overset{n \to \infty}{\to} \sin x = y \quad \square$$