Gewöhnliche Differentialgleichungen
FSU Jena - SS 2007
Serie 03 - Lösungen

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Aufgabe 01

a)

$$y' + y\tan x = 0 \;\Rightarrow\; SL: y \equiv 0$$

$$Für\ y \neq 0 : \ln|y| = \int \frac{dy}{y} = -\int \tan x\; dx = \ln|\cos x| + C,\; C \in \mathbb{R},\; x \neq \frac{\pi}{2}(2k+1),\; k \in \mathbb{Z}$$

$$\Rightarrow\; y(x) = A\cos x,\; A \in \mathbb{R}\setminus\{0\}$$

b)

$$y' + \frac{y}{x} = x^3,\; x \neq 0 \;\rightarrow\; Homogene:\; y' = -\frac{y}{x} \;\Rightarrow\; y_h = \frac{A}{x},\; A \in \mathbb{R}$$

$$Ansatz:\; y_p(x) = \frac{v(x)}{x} \;\Rightarrow\; y' + \frac{y}{x} = \frac{v'}{x} = x^3 \;\Rightarrow\; v' = x^4 \;\Rightarrow\; v = \frac{x^5}{5} \;\Rightarrow\; y_p(x) = \frac{x^4}{5}$$

$$\Rightarrow\; Allgemeine:\; y(x) = \frac{A}{x} + \frac{x^4}{5},\; A \in \mathbb{R}$$

c)

$$(1+x^2)y' - 2xy = (1+x^2)^2 \;\Rightarrow\; y' - \frac{2x}{1+x^2}y = 1+x^2 \;\rightarrow\; Homogene:\; y' = \frac{2x}{1+x^2}y \;\Rightarrow\; y_h = A(1+x^2)\; A \in \mathbb{R}$$

$$Ansatz:\; y_p(x) = u(x)\cdot(1+x^2) \;\Rightarrow\; y' = u'(1+x^2) + 2ux \;\Rightarrow\; u' = 1 \;\Rightarrow\; u = x$$

$$\Rightarrow\; y_p(x) = x(1+x^2) \;\Rightarrow\; Allgemeine:\; y(x) = (1+x^2)\cdot(A+x),\; A \in \mathbb{R}$$

d)

$$(xy'-1)\ln x = 2y \;\Rightarrow\; y' - \frac{2y}{x\ln x} = \frac{1}{x},\; 1 \neq x > 0 \;\rightarrow\; Homogene:\; y' = \frac{2y}{x\ln x} \;\Rightarrow\; y_h = A\cdot\ln^2 x,\; A \in \mathbb{R}$$

$$Ansatz:\; y_p(x) = u(x)\cdot\ln^2 x \;\Rightarrow\; u' = \frac{1}{x\ln^2 x} \;\Rightarrow\; u = -\frac{1}{\ln x} \;\Rightarrow\; y_p = -\ln x$$

$$\Rightarrow\; Allgemeine:\; y(x) = A\cdot\ln^2 x - \ln x,\; A \in \mathbb{R},\; x > 0$$

e)

$$xy' + y = x^3 + 3x + 2 \overset{x\neq 0}{\Rightarrow} y' + \frac{y}{x} = x + 3 + \frac{2}{x} \;\rightarrow\; Homogene:\; y' = -\frac{y}{x} \;\Rightarrow\; y_h(x) = \frac{A}{x},\; A \in \mathbb{R}$$

$$Ansatz:\; y_p(x) = \frac{u(x)}{x} \;\Rightarrow\; u' = x^2 + 3x + 2 \;\Rightarrow\; u = \frac{x^3}{3} + 3\frac{x^2}{2} + 2x \;\Rightarrow\; y_p(x) = \frac{x^2}{3} + \frac{3x}{2} + 2$$

$$\Rightarrow\; Allgemeine:\; y(x) = \frac{A}{x} + \frac{x^2}{3} + \frac{3x}{2} + 2,\; A \in \mathbb{R},\; x \neq 0$$

f)

$$y' + \frac{3x^2+1}{x(x^2+1)}y = \frac{1}{\sqrt{x^2+1}},\; x \neq 0 \;\rightarrow\; Homogene:\; y' = -\frac{3x^2+1}{x(x^2+1)}y \;\Rightarrow\; y_h(x) = \frac{A}{x(x^2+1)},\; A \in \mathbb{R}$$

$$Ansatz:\; y_p(x) = \frac{u(x)}{x(x^2+1)} \;\Rightarrow\; \frac{u'}{x^3+x} = \frac{1}{\sqrt{x^2+1}} \;\Rightarrow\; u' = x\sqrt{x^2+1} \;\Rightarrow\; u = \frac{(x^2+1)^{3/2}}{3}$$

$$Allgemeine:\; y(x) = \frac{A}{x(x^2+1)} + \frac{\sqrt{x^2+1}}{3x},\; A \in \mathbb{R},\; x \neq 0$$

Aufgabe 02

a)

$$x^2y' + y - 2xy - x^2 = 0 \overset{x\neq 0}{\Rightarrow} y' + y\frac{(1-2x)}{x^2} = 1 \;\rightarrow\; Homogene: y' = -y\frac{(1-2x)}{x^2} \;\Rightarrow\; y_h(x) = Ax^2e^{\frac{1}{x}},\; A \in \mathbb{R}$$

$$Ansatz:\; y(x) = u(x)\cdot x^2e^{1/x} \;\Rightarrow\; u' = \frac{e^{-1/x}}{x^2} \;\Rightarrow\; u = e^{-1/x} + A,\; A \in \mathbb{R} \;\Rightarrow\; Allgemeine:\; y(x) = Ax^2e^{1/x} + x^2$$

$$AWP:\; y(1) = 1 \;\Rightarrow\; y = x^2,\; x \neq 0$$

b)

$$x(y' - y) = (1+x^2)e^x \overset{x\neq 0}{\Rightarrow} y' - y = \frac{1+x^2}{x}\cdot e^x \;\rightarrow\; Homogene:\; y' = y \;\Rightarrow\; y = Ae^x,\; A \in \mathbb{R}$$

$$Ansatz:\; y = u(x)\cdot e^x \;\Rightarrow\; u' = \frac{1+x^2}{x} \;\Rightarrow\; u = \ln|x| + \frac{x^2}{2} \;\Rightarrow\; Allgemeine:\; y(x) = \left(\ln|x| + \frac{x^2}{2} + A\right)\cdot e^x$$

$$AWP:\; y(-1) \overset{!}{=} 1/e \;\Rightarrow\; y(x) = \left(\ln|x| + \frac{x^2}{2} + \frac{1}{2}\right)\cdot e^x,\; x \neq 0$$

c)

$$y' - y\sin x = \sin 2x \;\rightarrow\; Homogene:\; y' = y\sin x \;\Rightarrow\; y_h(x) = Ae^{-\cos x},\; A \in \mathbb{R}$$

$$Ansatz:\; y(x) = u(x)\cdot e^{-\cos x} \;\Rightarrow\; u' = \sin 2x \cdot e^{\cos x} \;\Rightarrow\; u = 2e^{\cos x}\cdot(1-\cos x) + A,\; A \in \mathbb{R}$$

$$\Rightarrow\; Allgemeine:\; y(x) = 2(1-\cos x) + Ae^{-\cos x},\; AWP:\; y(\pi/2) \overset{!}{=} 3 \;\Rightarrow\; y(x) = 2(1-\cos x) + e^{-\cos x},\; x \in \mathbb{R}$$

d)

$$y' + \frac{2x-1}{x(x-1)}y = \frac{1}{x(x-1)},\ 0 \neq x \neq 1\ \rightarrow\ Homogene:\ y' = -\frac{2x-1}{x(x-1)}y\ \Rightarrow\ y_h(x) = \frac{A}{x(x-1)},\ A \in \mathbb{R}$$

$$Ansatz:\ y_h(x) = \frac{u(x)}{x(x-1)}\ \Rightarrow\ u' = 1\ \Rightarrow\ u = x + C,\ C \in \mathbb{R}\ \Rightarrow\ Allgemeine:\ y(x) = \frac{x+C}{x(x-1)}$$

$$AWP:\ y(1/2) \overset{!}{=} -1\ \Rightarrow\ y(x) = \frac{4x-1}{4x(x-1)},\ 0 \neq x \neq 1$$

Aufgabe 03

a)

$$xy' + 2y - xy^2 = 0 \overset{x\neq 0}{\Rightarrow} y' + \frac{2}{x}y - y^2 = 0,\ z := \frac{1}{y},\ y \neq 0\ \Rightarrow\ y' = -\frac{z'}{z^2} = \frac{1}{z^2} - \frac{2}{xz}\ \Rightarrow\ z' - \frac{2}{x}z = -1$$

$$Homogene: z' = \frac{2}{x}z\ \Rightarrow\ z_h = A \cdot x^2,\ A \in \mathbb{R} \overset{VDK}{\rightarrow}\ z_p = u(x) \cdot x^2\ \rightarrow\ u' = -\frac{1}{x^2}\ \Rightarrow\ u = \frac{1}{x}\ \rightarrow z = x + Ax^2,\ A \in \mathbb{R}$$

$$\Rightarrow\ y = \frac{1}{x + Ax^2},\ 0 \neq x \neq \frac{-1}{A} \overset{AWP}{\Rightarrow}\ y(1) = 1\ \Rightarrow\ y = \frac{1}{x},\ x \neq 0$$

b)

$$y' - \frac{3x^2+1}{x(x^2+1)}y = -\frac{y^2}{\sqrt{x^2+1}},\ x \neq 0,\ z := \frac{1}{y},\ y \neq 0\ \Rightarrow\ z' = \frac{1}{\sqrt{x^2+1}} - \frac{3x^2+1}{x(x^2+1)} \cdot z$$

$$\rightarrow\ Homogene:\ z' = -\frac{3x^2+1}{x(x^2+1)} \cdot z\ \rightarrow\ z_h = \frac{A}{x(x^2+1)},\ A \in \mathbb{R},\ z_p = \frac{u(x)}{x(x^2+1)}\ \Rightarrow\ u' = x\sqrt{x^2+1}$$

$$\Rightarrow\ u = \frac{1}{3} \cdot (x^2+1)^{3/2}\ \Rightarrow\ z = \frac{(x^2+1)^{3/2} + 3A}{3x(x^2+1)} \overset{AWP}{\Rightarrow}\ y = \frac{3x}{\sqrt{x^2+1}},\ x \neq 0$$

c)

$$xy' - y = y^2 \cos x \overset{x\neq 0}{\Rightarrow} y' = \frac{y}{x} + \frac{y^2}{x}\cos x,\ z := \frac{1}{y},\ y \neq 0\ \Rightarrow\ z' = -\frac{z}{x} - \frac{\cos x}{x}$$

$$\rightarrow\ Homogene:\ z' = -\frac{z}{x}\ \Rightarrow\ z_h = \frac{A}{x},\ A \in \mathbb{R} \overset{VDK}{\rightarrow} y_p = \frac{u(x)}{x}\ \Rightarrow\ u' = -\cos x\ \Rightarrow\ u = -\sin x\ \Rightarrow\ z = \frac{A - \sin x}{x}$$

$$\overset{AWP}{\Rightarrow}\ y = \frac{x}{1 - \sin x},\ x \neq \frac{\pi}{2} + 2\pi k,\ k \in \mathbb{Z}$$

d)

$$y' + \frac{1}{x}y = xy^2,\ x \neq 0,\ z := \frac{1}{y} \Rightarrow\ z' = \frac{z}{x} - x\ \rightarrow\ Homogene:\ z' = \frac{z}{x}\ \Rightarrow\ z = A \cdot x,\ A \in \mathbb{R}$$

$$\stackrel{VDK}{\Rightarrow}\ z_p = u(x) \cdot x\ \Rightarrow\ u' = -1\ \Rightarrow\ u = -x\ \Rightarrow\ z = (A - x) \cdot x,\ A \in \mathbb{R}\ \Rightarrow\ y = \frac{1}{x(A-x)},\ x \neq A$$

$$AWP:\ y(1) = 1\ \Rightarrow\ y = \frac{1}{x(2-x)},\ 0 \neq x \neq 2$$

e)

$$2xy' - y - 10x^3y^5 = 0\ \stackrel{x \neq 0}{\Rightarrow}\ y' - \frac{y}{2x} - 5x^2y^5 = 0,\ z := \frac{1}{y^4}\ \Rightarrow\ z' = -\frac{2z}{x} - 20x^2$$

$$\rightarrow\ Homogene:\ z' = -\frac{2z}{x}\ \Rightarrow\ z_h = \frac{A}{x^2},\ A \in \mathbb{R}\ \stackrel{VDK}{\rightarrow}\ z_p = \frac{u(x)}{x^2}\ \Rightarrow\ u' = -20x^4\ \Rightarrow\ u = -4x^5\ \Rightarrow\ z = \frac{A - 4x^5}{x^2}$$

$$AWP:\ y(2) = 1/2\ \Rightarrow\ y = \sqrt[4]{\frac{x^2}{3 \cdot 2^6 - 4x^5}},\ x < \sqrt[5]{48}$$

Aufgabe 04

a)

$$y' = y^2 - \frac{2}{x^2},\ y_p = \frac{1}{x},\ x \neq 0,\ Ansatz:\ y = y_p + \frac{1}{z}\ \rightarrow\ y' = -\frac{1}{x^2} - \frac{z'}{z^2} = \left(\frac{1}{x} + \frac{1}{z}\right)^2 - \frac{2}{x^2}\ \Rightarrow\ z' = -\frac{2z}{x} - 1$$

$$\rightarrow\ Homogene:\ z' = -\frac{2z}{x}\ \Rightarrow\ z_h = \frac{A}{x^2},\ A \in \mathbb{R}\ \stackrel{VDK}{\rightarrow}\ z_p = \frac{u(x)}{x^2}\ \Rightarrow\ u' = -x^2\ \Rightarrow\ u = -\frac{x^3}{3}\ \Rightarrow\ z = \frac{C - x^3}{3x^2},\ C \in \mathbb{R}$$

$$\Rightarrow\ y = \frac{1}{x} + \frac{3x^2}{C - x^3},\ 0 \neq x \neq \sqrt[3]{C},\ AWP:\ y(1) = -2\ \Rightarrow\ y = -\frac{2}{x},\ x \neq 0$$

b)

$$x(x-1)y' - (1+2x)y + y^2 + 2x = 0\ \stackrel{x \neq 0}{\Rightarrow}\ y' = \frac{-2x}{x(x-1)} + \frac{(1+2x)}{x(x-1)}y - y^2\ \rightarrow\ y_p \equiv 1\ \rightarrow\ Ansatz:\ y = y_p + \frac{1}{z}$$

$$\Rightarrow\ y' = -\frac{z'}{z^2} = \frac{-2x}{x(x-1)} = \frac{(1+2x)}{x(x-1)} \cdot \left(1 + \frac{1}{z}\right) - \frac{1}{x(x-1)} \cdot \left(1 + \frac{1}{z}\right)^2\ \Rightarrow\ z' + \frac{(2x-1)}{x(x-1)}z = \frac{1}{x(x-1)}$$

$$\stackrel{2d}{\Rightarrow}\ z = \frac{x + C}{x(x-1)}\ \stackrel{AWP}{\Rightarrow}\ y = \frac{4x^2 - 1}{4x(x-1)},\ 0 \neq x \neq 1$$

c)

$$y' = 2 - 2xy + y^2 \;\rightarrow\; y_p = 2x \; Ansatz: \; y = y_p + \frac{1}{z} \;\Rightarrow\; y' = 2 - \frac{z'}{z^2} = 2 - 2x\left(2x + \frac{1}{z}\right) + \left(2x + \frac{1}{z}\right)^2 \;\Rightarrow\; z' = -2xz - 1$$

$$\rightarrow\; Homogene: \; z' = -2xz \;\Rightarrow\; z_h = Ae^{-x^2}, \; A \in \mathbb{R} \;\overset{VDK}{\rightarrow}\; Ansatz: \; z = u(x) \cdot e^{-x^2} \;\Rightarrow\; u' = -e^{x^2}$$

$$\Rightarrow\; u = -\int_0^x e^{t^2} dt \;\Rightarrow\; z = e^{-x^2} \cdot \left(C - \int_0^x e^{t^2} dt\right), \; C \in \mathbb{R} \;\overset{AWP}{\Rightarrow}\; y = 2x + \frac{e^{x^2}}{1 - \int_0^x e^{t^2} dt}, \; x \in \mathbb{R} \setminus \{\varphi\} : \int_0^{\varphi} e^{t^2} dt = 1$$