Gewöhnliche Differentialgleichungen - Übungsserie 1
FSU Jena - SS 07
- Lösungen -

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***Wenn nichts anderes gesagt wird, sind C eine beliebige Konstante und (x_0, y_0) der Anfangswert-Punkt**

Aufgabe 01

a)

$$y' = (1+x^2)^{-1} \ \Rightarrow \ y(x) = \int \frac{dx}{1+x^2} = \arctan(x) + C, \ x \in \mathbb{R}$$

b)

$$y' = (1-x^2)^{-1} \ \Rightarrow \ y(x) = \int \frac{dx}{1-x^2} = \frac{1}{2}\int \left[\frac{1}{1-x} + \frac{1}{1+x}\right] dx = \frac{1}{2} \cdot [\ln(1+x) - \ln(1-x)] + C$$

$$= \frac{1}{2} \cdot \ln\left(\frac{1+x}{1-x}\right) + C, \ x \in (-1, 1)$$

c)

$$y' = (1+y^2) \ \Rightarrow \ \arctan(y) = \int \frac{dy}{1+y^2} = \int dx = x + C \ \Rightarrow \ y(x) = \tan(x+C), \ x \in \mathbb{R}$$

d)

$$y' = 1 - y^2 \ \Rightarrow \ \frac{1}{2} \cdot \ln\left|\frac{y+1}{y-1}\right| = \int \frac{dy}{1-y^2} = \int dx = x + \frac{C}{2} \ \Rightarrow \ y(x) = \frac{1+e^{2x+C}}{e^{2x+C}-1}, \ x \neq -\frac{C}{2}$$

$$Sonderlösung: \ y(x) = \pm 1 : const$$

Aufgabe 2

$$y' = -\frac{\sqrt{a^2-x^2}}{x} \ \Rightarrow \ y = -\int \frac{\sqrt{a^2-x^2}}{x} dx$$

$$Sub: \ x := a\sin t \ \rightarrow \ y = -\int \frac{a\cos t\sqrt{1-\sin^2 t}}{\sin t} dt = -a\int \frac{\cos^2 t}{\sin t} dt = a\int \left[-\frac{1}{\sin t} + \sin t\right] dt$$

$$= -a\ln\left|\tan\left(\frac{t}{2}\right)\right| - a\cos t + C, \quad x = a \ \rightarrow \ t = \frac{\pi}{2} \ \Rightarrow \ y(a) = 0 + 0 + C \stackrel{!}{=} 0 \ \Rightarrow \ C = 0$$

Aufgabe 3

$$P' = \lambda P(K-P) \;\Rightarrow\; \lambda t = \int \lambda dt = \int \frac{dP}{P(K-P)} = \frac{1}{K} \cdot \int \left[\frac{1}{P} + \frac{1}{K-P}\right] dt = \frac{1}{K} \cdot [\ln(P) - \ln(K-P)] - \frac{C}{K}$$

$$= \frac{1}{K} \cdot \ln\left(\frac{P}{K-P}\right) - \frac{C}{K} \;\Rightarrow\; P(t) = \frac{Ke^{K\lambda t + C}}{1 + e^{K\lambda t + C}}$$

$$Sonderlösungen: \; P(x) = K : const \;\vee\; P(x) = 0 : const$$

Aufgabe 4

Sei $S(t)$ die Salzmenge die im Wasser im Zeitpunkt t (in Minuten gemessen) vorhanden ist. Dann läuft pro Minute $2S(t)/V_0$ Menge Salz raus, wobei $V_0 := 10^3\ L$ das konstante Volumen der Lösung ist. Demzufolge gilt:

$$\frac{dS}{dt} = -\frac{2S}{V_0} \;\Rightarrow\; \ln(S) = \int \frac{dS}{S} = -\int \frac{2dt}{V_0} = -\frac{2t}{V_0} + \ln(C) \;\Rightarrow\; S(t) = Ce^{-2t/V_0}$$

Wir wissen außerdem dass $S(0) = 50Kg$. Folglich:

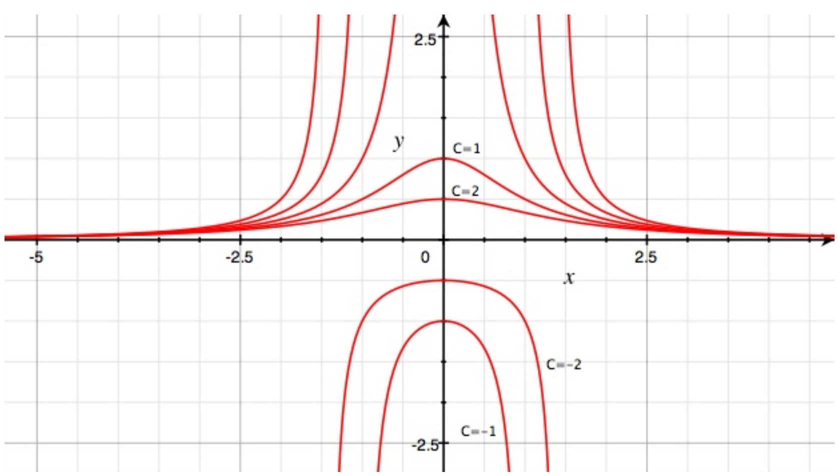
$$S(0) = C = 50Kg \;\Rightarrow\; S(t) = 50 \cdot e^{-2t/V_0} = 50 \cdot e^{-t/500} Kg$$

Aufgabe 5

a)

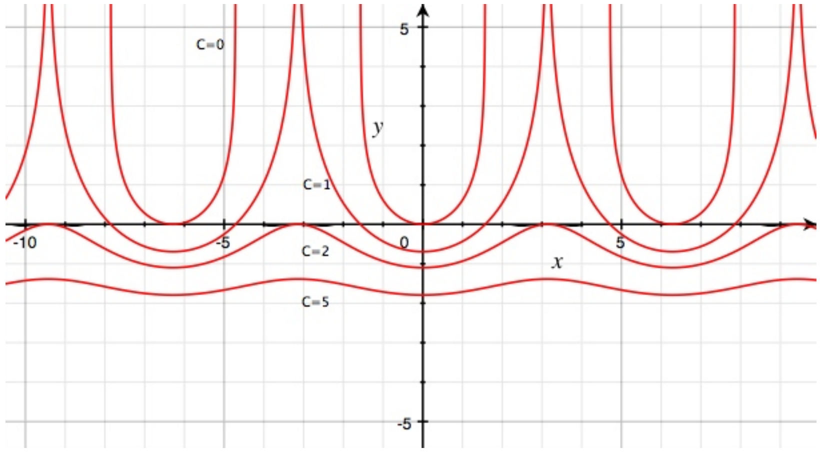
$$y' = -2xy^2 \;\Rightarrow\; \frac{1}{y} = \int -\frac{dy}{y^2} = \int 2x dx = x^2 + C \;\Rightarrow\; y(x) = \frac{1}{x^2 + C},\; x \neq \sqrt{-C}$$

$$Sonderlösung: \; y(x) = 0 : const$$



b)

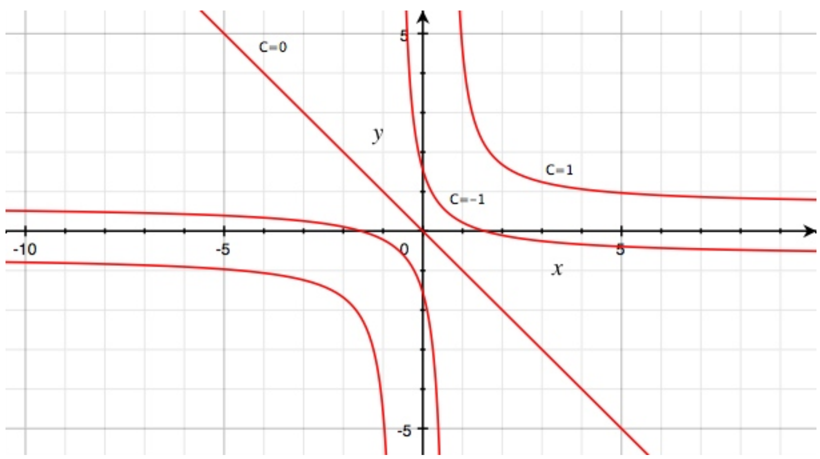
$$y' = e^y \sin x \;\Rightarrow\; -e^{-y} = \int \frac{dy}{e^y} = \int \sin x dx = -\cos x - C \;\Rightarrow\; y(x) = -\ln[\cos x + C],\; x \in \{x \in \mathbb{R} : \cos x + C > 0\}$$



c)

$$(x^2+1)y' = -y^2 - 1 \;\Rightarrow\; \arctan y = \int \frac{dy}{y^2+1} = \int -\frac{dx}{x^2+1} = -\arctan x - C$$

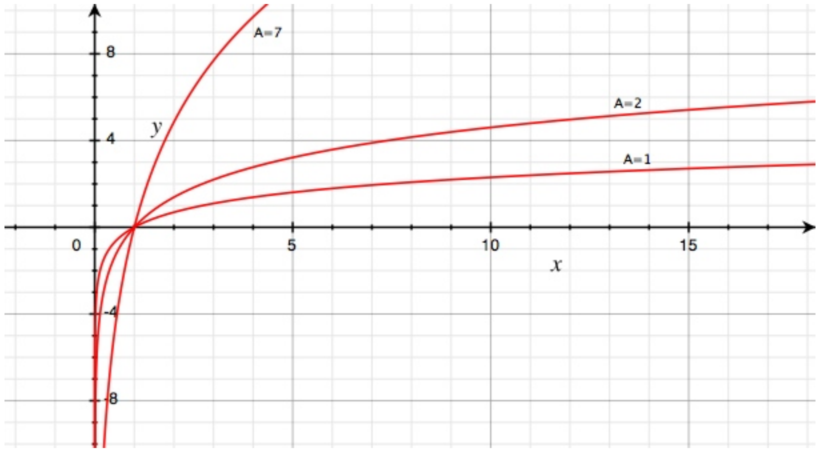
$$\Rightarrow\; y(x) = \tan\left(-\arctan x - C\right) = -\tan\left(\arctan x + C\right),\;\; x \in \mathbb{R}$$



d)

$$y = y' x \ln x \;\Rightarrow\; \ln|y| = \int \frac{dy}{y} = \int \frac{dx}{x\ln x} = \ln|\ln x| + C \;\Rightarrow\; y(x) = A\ln x,\;\; x>0,\;\; A \in \mathbb{R}\setminus\{0\} : const$$

$$Sonderlösung:\;\; y(x) = 0 : const$$



Aufgabe 6

a)

$$y^3y' = x^2 \;\Rightarrow\; \frac{1}{4}\cdot\left(y^4 - y_0^4\right) = \int_{y_0}^{y} t^3 dt = \int_{x_0}^{x} t^2 dt = \frac{1}{3}\cdot\left(x^3 - x_0^3\right) \;\Rightarrow\; y^4 = \frac{4}{3}\cdot\left(x^3 - 15\right),\; x \geq \sqrt[3]{15}$$

b)

$$y' = \frac{e^{-y^2}}{y(2x+x^2)} \;\Rightarrow\; \frac{1}{2}\cdot\left[e^{y^2} - e^{y_0^2}\right] = \int_{y_0}^{y} te^{t^2} dt = \int_{x_0}^{x} \frac{dt}{2t+t^2} = \int_{x_0}^{x} \frac{1}{2}\left[\frac{1}{t} - \frac{1}{2+t}\right] = \frac{1}{2}\cdot\left[\ln\left|\frac{x}{2+x}\right| - \ln\left(\frac{x_0}{2+x_0}\right)\right]$$

$$\Rightarrow\; y^2 = \ln\left[\ln\left|\frac{2x}{2+x}\right| + e\right]$$

$$\ln\left|\frac{2x}{2+x}\right| \overset{!}{>} -e \;\wedge\; \frac{2x}{2+x} \overset{!}{\neq} 0 \;\Rightarrow\;,\; x \in (-\infty,-2)\cup(-2,0)\cup\left(\frac{2}{2e^e-1},\infty\right)$$

c)

$$xyy' = 1 - x^2 \;\Rightarrow\; \frac{1}{2}\cdot\left[y^2 - y_0^2\right] = \int_{y_0}^{y} t dt = \int_{x_0}^{x} \frac{1-t^2}{t} dt = \int_{x_0}^{x}\left[\frac{1}{t} - t\right] dt = \left[\ln t - \frac{t^2}{2}\right]_{x_0}^{x} = \ln\left[\frac{x}{x_0}\right] + \frac{x_0^2 - x^2}{2}$$

$$\Rightarrow\; y^2 = 2\ln x + 5 - x^2,\; x > 0$$

d)

$$xy' = y^2 - y \;\Rightarrow\; \ln\left|\frac{y-1}{y}\right| + \ln\left|\frac{y_0}{y_0-1}\right| = \int_{y_0}^{y}\left[\frac{1}{t-1} + \frac{1}{t}\right] dt = \int_{y_0}^{y} \frac{dt}{t^2-t} = \int_{x_0}^{x} \frac{dt}{t} = \ln\left|\frac{x}{x_0}\right|$$

$$\Rightarrow\; \frac{y-1}{y} = x \;\Rightarrow\; y(x) = \frac{1}{1-x},\; x \neq 1$$

e)

$$y'\tan x - y = 1 \;\Rightarrow\; \ln\left|\frac{1+y}{1+y_0}\right| = \int_{y_0}^{y} \frac{t}{1+t} = \int_{x_0}^{x} \frac{dt}{\tan t} = \ln\left|\frac{\sin x}{\sin x_0}\right|$$

$$\Rightarrow\; y(x) = \frac{1+y_0}{\sin x_0}\cdot\sin x - 1 = 2\sin x - 1,\; x \neq \frac{\pi}{2} + k\pi,\; k \in \mathbb{Z}$$

f)

$$y' = 1 - y^2, \quad y(0) = 1$$

$$Ansatz: \; y(x) = 1 \; \forall \; x \in \mathbb{R}$$

$$Probe: \; y' = 0 = 1 - y^2, \quad y(0) = 1 \; \rightarrow \; Lösung$$

$$y(x) = -1 : const \; : \; Keine \; Lösung$$

$$Für \; 1 - y^2 \neq 0 : \int \frac{dy}{1 - y^2} = \int dx \; \Rightarrow \; \ln \left| \frac{y+1}{y-1} \right| = 2x + C \; \rightarrow \; Invalid \; für \; y_0 = 1$$

g)

$$y' = \cos^2 y - 1, \quad y(\pi) = 2\pi$$

$$Ansatz: \; y(x) = 2\pi \; \forall \; x \in \mathbb{R}$$

$$Probe: \; y' = 0 = \cos^2 2\pi - 1 = \cos^2 y - 1, \quad y(\pi) = 2\pi \; \rightarrow \; Lösung$$

$$Für \; \cos^2 y - 1 \neq 0 : \cot y = \int \frac{dy}{\sin^2 y} = \int \frac{dy}{\cos^2 y - 1} = \int dx = x + C \; \rightarrow \; keine \; Lösung \; für \; y_0 = 2\pi$$