Theoretische Elektrodynamik
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Serie 01 - Lösungen

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Aufgabe 01

Bemerkung: $A, B, C \in \mathbb{R}^3$

a)

$$A \times (B \times C) = A_i (B \times C)_j \cdot \varepsilon_{ijk} \cdot \vec{e}_k = A_i B_l C_m \varepsilon_{lmj} \varepsilon_{kij} \cdot \vec{e}_k = A_i B_l C_m \cdot (\delta_{lk}\delta_{mi} - \delta_{li}\delta_{mk}) \cdot \vec{e}_k$$

$$= A_i C_m \delta_{lk}\delta_{mi} \cdot B_l \cdot \vec{e}_k - A_i B_l \delta_{li}\delta_{mk} \cdot C_m \cdot \vec{e}_k = A_i C_i \delta_{lk} \cdot B_l \cdot \vec{e}_k - A_i B_i \delta_{mk} \cdot C_m \cdot \vec{e}_k$$

$$= A_i C_i \cdot B_l \vec{e}_l - A_i B_i \cdot C_m \vec{e}_m = (A \cdot C) B - (A \cdot B) C \quad \square$$

b)

$$[A \cdot (B \times C)] = A_i (B \times C)_i = A_i B_j C_k \varepsilon_{jki} = B_j C_k A_i \varepsilon_{kij} = B_j (C_k \times A_i)_j = B \cdot (C \times A)$$

$$\Rightarrow \; B \cdot (C \times A) = C \cdot (A \times B) \quad \square$$

c)

$$(A \cdot \nabla) B + (B \cdot \nabla) A + A \times \operatorname{rot} B + B \times \operatorname{rot} A = \left(A_i \frac{\partial}{\partial x_i} \right) B + \left(B_i \frac{\partial}{\partial x_i} \right) A + A_i (\operatorname{rot} B)_j \varepsilon_{ijk} \cdot \vec{e}_k + B_i (\operatorname{rot} A)_j \varepsilon_{ijk} \cdot \vec{e}_k$$

$$= A_i \frac{\partial B_j}{\partial x_i} \cdot \vec{e}_j + B_i \frac{\partial A_j}{\partial x_i} \cdot \vec{e}_j + A_i \frac{\partial B_m}{\partial x_l} \varepsilon_{lmj} \underbrace{\varepsilon_{ijk}}_{\varepsilon_{kij}} \cdot \vec{e}_k + B_i \frac{\partial A_m}{\partial x_l} \varepsilon_{lmj} \underbrace{\varepsilon_{ijk}}_{\varepsilon_{kij}} \cdot \vec{e}_k$$

$$= A_i \frac{\partial B_j}{\partial x_i} \cdot \vec{e}_j + B_i \frac{\partial A_j}{\partial x_i} \cdot \vec{e}_j + A_i \frac{\partial B_m}{\partial x_l} (\delta_{lk}\delta_{mi} - \delta_{li}\delta_{mk}) \cdot \vec{e}_k + B_i \frac{\partial A_m}{\partial x_l} (\delta_{lk}\delta_{mi} - \delta_{mk}\delta_{li}) \cdot \vec{e}_k$$

$$= A_i \frac{\partial B_j}{\partial x_i} \cdot \vec{e}_j + B_i \frac{\partial A_j}{\partial x_i} \cdot \vec{e}_j + A_i \frac{\partial B_i}{\partial x_k} \cdot \vec{e}_k - A_i \frac{\partial B_k}{\partial x_i} \cdot \vec{e}_k + B_i \frac{\partial A_i}{\partial x_k} \cdot \vec{e}_k - B_i \frac{\partial A_k}{\partial x_i} \cdot \vec{e}_k$$

$$= A_i \frac{\partial B_i}{\partial x_k} \cdot \vec{e}_k + B_i \frac{\partial A_i}{\partial x_k} \cdot \vec{e}_k = \frac{\partial}{\partial x_k} (A_i B_i) \cdot \vec{e}_k = \frac{\partial}{\partial x_k} (A \cdot B) \cdot \vec{e}_k = \nabla (A \cdot B) \quad \square$$

d)

$$\lambda(B \cdot \operatorname{rot} A - A \cdot \operatorname{rot} B) + A \times B \cdot \operatorname{grad} \lambda = \lambda \left[B_i (\operatorname{rot} A)_i - A_j (\operatorname{rot} B)_j\right] + A_j B_i \varepsilon_{jik} (\operatorname{grad} \lambda)_k$$

$$= \lambda \left[B_i \frac{\partial A_j}{\partial x_k} \underbrace{\varepsilon_{kji}}_{\varepsilon_{jik}} - A_j \frac{\partial B_i}{\partial x_k} \underbrace{\varepsilon_{kij}}_{-\varepsilon_{jik}} \right] + A_j B_i \varepsilon_{jik} \frac{\partial \lambda}{\partial x_k} = \frac{\partial}{\partial x_k} \left[\lambda A_i B_j \varepsilon_{ijk}\right] = \frac{\partial}{\partial x_k} (\lambda A \times B)_k = \operatorname{div}(\lambda A \times B) \quad \square$$

Aufgabe 02

$$\operatorname{div}(A \times B) \overset{1d}{=} B \cdot \underbrace{\operatorname{rot} A}_{0} - A \cdot \underbrace{\operatorname{rot} B}_{0} + A \times B \cdot \underbrace{\operatorname{grad} 1}_{0} = 0$$

$$\operatorname{rot}(A \times B) = \frac{\partial}{\partial x_i} (A \times B)_j \varepsilon_{ijk} \cdot \vec{e}_k = \frac{\partial}{\partial x_i} A_l B_m \varepsilon_{lmj} \underbrace{\varepsilon_{ijk}}_{\varepsilon_{kij}} \cdot \vec{e}_k = \frac{\partial}{\partial x_i} A_l B_m (\delta_{lk}\delta_{mi} - \delta_{li}\delta_{mk}) \cdot \vec{e}_k$$

$$= \frac{\partial}{\partial x_i} (A_k B_i - A_i B_k) \cdot \vec{e}_k = \left(\frac{\partial A_k}{\partial x_i} B_i + \frac{\partial B_i}{\partial x_i} A_k - \frac{\partial A_i}{\partial x_i} B_k - \frac{\partial B_k}{\partial x_i} A_i \right) \cdot \vec{e}_k$$

$$= (B \cdot \nabla) A + (\operatorname{div} B) \cdot A - (\operatorname{div} A) \cdot B - (A \cdot \nabla) B = (B \cdot \nabla) A - (A \cdot \nabla) B$$

Aufgabe 03

a)

$$\vec{A} = ax\vec{e}_x + by\vec{e}_y + cz\vec{e}_z, \;\; K = \left\{ \vec{r} : x^2 + y^2 + z^2 \leq R^2 \right\}$$

$$LHS: \;\; \int_{\partial K} \vec{A} \cdot d\vec{f} = \int_{\partial K} \vec{A} \cdot \vec{e}_\rho \; df = \frac{1}{R} \cdot \int_{\partial K} \left(ax^2 + by^2 + cz^2\right) \; df$$

$$= R^3 \cdot \int_{\partial K} (a \sin^2 \vartheta \cos^2 \varphi + b \sin^2 \vartheta \sin^2 \varphi + c \cos^2 \vartheta) \cdot \sin \vartheta \; d\vartheta \; d\varphi$$

$$= R^3 \cdot \int_0^{2\pi} \left\{ \int_0^{\pi} \left[\sin^2 \vartheta \left(a \cos^2 \varphi + b \sin^2 \varphi \right) + c \cos^2 \vartheta \sin \vartheta \right] \; d\vartheta \right\} \; d\varphi$$

$$= \frac{R^3}{3} \cdot \int_0^{2\pi} \left[4 \left(a \cos^2 \varphi + b \sin^2 \varphi \right) + 2c \right] \; d\vartheta = \frac{4\pi R^3}{3} \cdot (a + b + c)$$

$$RHS: \;\; \int_K \operatorname{div} \vec{A} \cdot dV = \int_K (a + b + c) \cdot dV = \frac{4\pi R^3}{3} \cdot (a + b + c) \quad \square$$

b)

$$\vec{r} \in \partial F,\ \vec{r} = 3R\cos t \cdot \vec{e}_x + 2R\sin t \cdot \vec{e}_y \ \rightarrow\ \dot{\vec{r}} = -3R\sin t \cdot \vec{e}_x + 2R\cos t \cdot \vec{e}_y$$

$$LHS:\ \int_{\partial F} \vec{A} \cdot d\vec{r} = \int_{\partial F} \vec{A} \cdot \dot{\vec{r}}\, dt = R^2 \cdot \int_0^{2\pi} \left(12\sin^2 t - 6\cos^2 t\right)\ dt$$

$$= R^2 \cdot \int_0^{2\pi} [6(1-\cos 2t) - 3(1+\cos 2t)]\ dt = 6\pi R^2$$

$$RHS:\ \int_F \operatorname{rot} \vec{A} \cdot d\vec{f} = \int_F \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial A_y}\right)\ df = \int_F df = 3R \cdot 2R \cdot \pi = 6\pi R^2$$