Differentialgeometrie auf Mannigfaltigkeiten
FSU Jena - WS 07/08
Serie 06 - Lösungen

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Aufgabe 1

Beginnen mit den Definitionen

$$\left(f \bigwedge g\right)(x_1, ..., x_{p+q}) := \frac{1}{p!q!} \cdot \sum_{\mathcal{P}\in S_{p+q}} \chi(\mathcal{P}) f\left(x_{\mathcal{P}(1)}, ..., x_{\mathcal{P}(p)}\right) \cdot g\left(x_{\mathcal{P}(p+1)}, ..., x_{\mathcal{P}(q)}\right)$$

und

$$a^{i_1} \wedge a^{i_2} \wedge ... \wedge a^{i_p}(x_1, ..., x_p) := \sum_{\mathcal{P}\in S_p} \chi(\mathcal{P}) \prod_{j=1}^{p} \left\langle x_j, a^{i_{\mathcal{P}(j)}}\right\rangle$$

und schreiben

$$\left\{a^1 \bigwedge \left(a^2 \wedge \dots a^p\right)\right\}(x_1, ..., x_p) = \frac{1}{(p-1)!} \sum_{\mathcal{P}\in S_p} \chi(\mathcal{P}) \left\langle a^1, x_{\mathcal{P}(1)}\right\rangle \cdot \left(a^2 \wedge \cdots \wedge a^p\right)\left(x_{\mathcal{P}(2)}, \dots, x_{\mathcal{P}(p)}\right)$$

$$= \frac{1}{(p-1)!} \sum_{\mathcal{P}\in S_p} \chi(\mathcal{P}) \left\langle a^1, x_{\mathcal{P}(1)}\right\rangle \cdot \sum_{\substack{\mathcal{Q}\in S_p \\ \mathcal{Q}(1)=1}} \chi(\mathcal{Q}) \prod_{i=2}^{p} \left\langle a^i, x_{\mathcal{P}(\mathcal{Q}(i))}\right\rangle$$

$$= \frac{1}{(p-1)!} \sum_{\mathcal{P}\in S_p} \chi(\mathcal{P}) \cdot \sum_{\substack{\mathcal{Q}\in S_p \\ \mathcal{Q}(1)=1}} \chi(\mathcal{Q}) \prod_{i=1}^{p} \left\langle a^i, x_{\mathcal{P}(\mathcal{Q}(i))}\right\rangle = \frac{1}{(p-1)!} \sum_{\mathcal{P}\in S_p} \sum_{\substack{\mathcal{Q}\in S_p \\ \mathcal{Q}(1)=1}} \chi(\mathcal{P})\chi(\mathcal{Q}) \prod_{i=1}^{p} \left\langle a^i, x_{\mathcal{P}(\mathcal{Q}(i))}\right\rangle$$

$$= \frac{1}{(p-1)!} \sum_{\substack{\mathcal{Q}\in S_p \\ \mathcal{Q}(1)=1}} \sum_{\mathcal{P}\in S_p} \chi(\mathcal{P}\circ\mathcal{Q}) \prod_{i=1}^{p} \left\langle a^i, x_{\mathcal{P}(\mathcal{Q}(i))}\right\rangle = \frac{1}{(p-1)!} \cdot \frac{1}{p} \sum_{l=1}^{p} \sum_{\substack{\mathcal{Q}\in S_p \\ \mathcal{Q}(1)=l}} \sum_{\mathcal{P}\in S_p} \chi(\mathcal{P}\circ\mathcal{Q}) \prod_{i=1}^{p} \left\langle a^i, x_{\mathcal{P}(\mathcal{Q}(i))}\right\rangle$$

$$= \frac{1}{p!} \sum_{\mathcal{Q}\in S_p} \sum_{\mathcal{P}\in S_p} \chi(\mathcal{P}\circ\mathcal{Q}) \prod_{i=1}^{p} \left\langle a^i, x_{\mathcal{P}(\mathcal{Q}(i))}\right\rangle = \frac{1}{p!} \sum_{\mathcal{R}\in S_p} \sum_{\substack{\mathcal{Q},\mathcal{P}\in S_p \\ \mathcal{P}\circ\mathcal{Q}=\mathcal{R}}} \chi(\mathcal{R}) \prod_{i=1}^{p} \left\langle a^i, x_{\mathcal{R}(i)}\right\rangle$$

$$= \frac{1}{p!} \sum_{\mathcal{R}\in S_p} \chi(\mathcal{R}) \prod_{i=1}^{p} \left\langle a^i, x_{\mathcal{R}(i)}\right\rangle \cdot \underbrace{\sum_{\substack{\mathcal{Q},\mathcal{P}\in S_p \\ \mathcal{P}\circ\mathcal{Q}=\mathcal{R}}} 1}_{p!} = \sum_{\mathcal{R}\in S_p} \chi(\mathcal{R}) \prod_{i=1}^{p} \left\langle a^i, x_{\mathcal{R}(i)}\right\rangle$$

$$= \left(a^1 \wedge \cdots \wedge a^p\right)(x_1, ..., x_p) \quad \square$$